

**VANDERBILT UNIVERSITY**  
**MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS**  
**HW 3**

Unless stated otherwise, the notation below is as in class.

1. PROBLEMS

**Problem 1.** Prove that harmonic functions are analytic.

**Problem 2.** Prove Liouville's theorem for harmonic functions in  $\mathbb{R}^n$ .

**Problem 3.** Prove Harnack's inequality for (non-negative) harmonic functions.

The remaining questions are about the heat equation in  $n$ -dimensions, i.e.,

$$u_t - \Delta u = 0 \text{ in } (0, \infty) \times \mathbb{R}^n. \quad (1.1)$$

**Problem 4.** Look for a solution to (1.1) in the form

$$u(t, x) = t^{-\alpha} v(t^{-\beta} x), \quad (1.2)$$

where  $\alpha$  and  $\beta$  will be chosen and  $v$  will be determined. More precisely, proceed as follows:

(a) Show that plugging (1.2) into (1.1) produces

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0, \quad (1.3)$$

where  $y := t^{-\beta} x$ .

(b) Set  $\beta = \frac{1}{2}$  in (1.3) to obtain

$$\Delta v(y) + \frac{1}{2} y \cdot \nabla v(y) + \alpha v(y) = 0. \quad (1.4)$$

(c) Assume that  $v$  is radially symmetric, i.e.,

$$v(y) = w(r), \quad (1.5)$$

where  $w$  is to be determined. Show that in this case (1.4) becomes

$$w'' + \frac{n-1}{r} w' + \frac{1}{2} r w' + \alpha w = 0. \quad (1.6)$$

(d) Set  $\alpha = \frac{n}{2}$  in (1.6) to find

$$(r^{n-1} w')' + \frac{1}{2} (r^n w)' = 0. \quad (1.7)$$

(e) From (1.7), conclude that

$$r^{n-1} w' + \frac{1}{2} r^n w = A, \quad (1.8)$$

where  $A$  is a constant.

(f) Set  $A = 0$  in (1.8) and conclude that

$$w(r) = B e^{-\frac{1}{4} r^2}, \quad (1.9)$$

where  $B$  is a constant.

(g) Combine (1.2), (1.5), (1.9), and take into account the choices of  $\alpha$  and  $\beta$ , to conclude that

$$u(t, x) = \frac{B}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad (1.10)$$

is a solution to (1.1).

**Problem 5.** Recall that

$$\Gamma(t, x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases}$$

is called the *fundamental solution of the heat equation*. Note that for  $t > 0$ ,  $\Gamma(t, x)$  is simply (1.10) with a specific choice of the constant  $B$ . In particular,  $\Gamma(t, x)$  is a solution of (1.1).

This choice of  $B$  is to guarantee  $\Gamma$  to integrate to 1, i.e., using the fact that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{\frac{n}{2}}, \quad (1.11)$$

show that for each  $t > 0$

$$\int_{\mathbb{R}^n} \Gamma(t, x) dx = 1.$$

(You do *not* have to show (1.11).)

**Problem 6.** Consider the initial-value problem for the heat equation:

$$u_t - \Delta u = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n, \quad (1.12a)$$

$$u(0, x) = g(x), \quad x \in \mathbb{R}^n. \quad (1.12b)$$

In (1.12), assume that  $g \in C^0(\mathbb{R}^n)$  and that there exists a constant  $C > 0$  such that  $|g(x)| \leq C$  for all  $x \in \mathbb{R}^n$ .

Recall that we showed existence of a solution by defining

$$u(t, x) := \int_{\mathbb{R}^n} \Gamma(t, x - y) g(y) dy, \quad t > 0, x \in \mathbb{R}^n. \quad (1.13)$$

Show that (1.13) is well-defined.

**Problem 7.** Provide the details of the proof given in class that  $u \in C^\infty((0, \infty) \times \mathbb{R}^n)$ , where  $u$  is defined by (1.13).

*Hint:* Use the following fact, that you do *not* need to prove. Let  $\alpha$  be a multiindex and  $t > 0$ . If

$$\int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x - y) g(y) dy$$

is well-defined, then

$$D^\alpha u(t, x) = \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x - y) g(y) dy,$$

where we write  $D_x^\alpha$  on the RHS to emphasize that the differentiation is with respect to the  $x$  variable.

**Problem 8.** Look up the mean value formula and the maximum principle for solutions to the heat equation.

## 2. SOLUTIONS

**Solution 1.** See section 2.2.3 of Evan's book.

**Solution 2.** See section 2.2.3 of Evan's book.

**Solution 3.** See section 2.2.3 of Evan's book.

**Solution 4.** These are a sequence of straightforward calculations that are done in the class notes.

**Solution 5.** Set  $z = x/\sqrt{4t}$  and change variables to find

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx = \int_{\mathbb{R}^n} e^{-|z|^2} (\sqrt{4t})^n dz = \pi^{\frac{n}{2}} (4t)^{\frac{n}{2}}.$$

**Solution 6.** We have

$$|u(t, x)| \leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dy.$$

Making the change of variables  $z = (y - x)/\sqrt{4t}$  we find

$$\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dy = (4t)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz < \infty.$$

**Solution 7.** Let  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$  be an arbitrary multiindex. Then

$$D_x^\alpha \Gamma(t, x - y) = \frac{p(t, x, y)}{t^M} e^{-\frac{|x-y|^2}{4t}}, \quad (2.1)$$

where  $M$  is a non-negative constant and  $p$  is a polynomial on its arguments (If (2.1) is not clear, take a few derivatives of  $\Gamma(t, x - y)$  and see the pattern that emerges.) Then, using the assumption on  $g$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x - y) g(y) dy \right| &\leq C \int_{\mathbb{R}^n} |D_x^\alpha \Gamma(t, x - y)| dy \\ &\leq C \int_{\mathbb{R}^n} \frac{|p(t, x, y)|}{t^M} e^{-\frac{|x-y|^2}{4t}} dy \\ &= \int_{\mathbb{R}^n} \frac{|q(t, x, z)|}{t^N} e^{-|z|^2} dz, \end{aligned}$$

where in the last step we changed variables  $z = (y - x)/\sqrt{4t}$ ,  $N$  is a non-negative constant, and  $q$  is polynomial on its arguments. We claim that there exists a constant  $C > 0$ , possibly depending on  $t$ , such that

$$\frac{|q(t, x, z)|}{t^N} e^{-|z|^2} \leq C e^{-\frac{1}{2}|z|^2}. \quad (2.2)$$

For, (2.2) is equivalent to

$$\frac{|q(t, x, z)|}{t^N} e^{-\frac{1}{2}|z|^2} \leq C. \quad (2.3)$$

For each fixed  $x$  and  $t > 0$ , the function  $\frac{|q(t, x, z)|}{t^N} e^{-\frac{1}{2}|z|^2}$  is a continuous function of  $z$ , and because the exponential decays faster than any polynomial, we conclude that  $\frac{|q(t, x, z)|}{t^N} e^{-\frac{1}{2}|z|^2}$  is bounded in  $\mathbb{R}^n$  as a function of  $z$  for each fixed  $x$  and  $t > 0$ , which is (2.3). Since the integral of  $e^{-\frac{1}{2}|z|^2}$  is finite, we have shown the result in view of the hint and the fact that  $\alpha$ ,  $x$ , and  $t > 0$  are arbitrary.

**Solution 8.** See sections 2.3.2 and 2.3.3 of Evan's book.