RECENT ADVANCES IN CLASSICAL AND RELATIVISTIC FLUIDS

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1. Introduction

In these lectures we will discuss:

- free-boundary classical(i.e. non-relativistic) fluids,
- relativistic fluids.

Each topic will contain sub-topics, each of which can be studied on its own and has recently witnessed many developments but it is of interest in thse lectures to consider the unifying aspects of thse subjects. We cite here two such aspects:

- techniques: the techniques we will employ in all problems share the feature that geometry plays a prominent role.
- modeling: we can see each problem to be studied as modeling one aspect of an "ultimate problem in the following sense.

Consider Einstein's equations (EE):

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta}.$$

Often in general relativity(GR) we study the vacuum EE ($T_{\alpha\beta} = 0$). This is important because it helps us understand the phenomena that are intrinsically due to gravity. Indeed a very rich and complex set of dynamic behaviors arise from the vacuum EE (See Jared and Stefano's lectures). But it is equally important to understand the case with matter ($T_{\alpha\beta} \neq 0$). In particular we can ask what can be said about the most "realistic" matter models. One such case comprises the study of stellar evolution. Typically one models a star as a compact fluid body with a dynamic interface separating the fluid region from a vacuum region. Thus, one wants to solve EE coupled to a fluid in the fluid region and vacuum EE outside the fluid region, with some appropriate boundary condition in the interface separating both regions. Such interface is not still but moves with the motion of the fluid, leading therefore, to a free-boundary problem. Moreover, stars undergo many extreme physical processes, so we can expect the formation of shocks in the fluid. Furthermore, we would like our fluid model to include viscosity since real fluids do have viscosity.

We are very far from understanding the scenario outlined above, i.e., GR + free boundary + shocks + viscous fluids. However we can try to understand each such topic separately, hoping to bring them together in some distant future. That is the motivation for these lectures.

1.1. **Notation and conventions.** We will use standard notation for function spaces and their norms. The (L^2-based) Sobolev spaces will be denoted $H^S(\mathbb{R}^4)$, $H^S(\Omega)$, etc. Many times we omit the function space argument (e.g. L^∞ stands for $L^\infty(\Omega)$, etc.) The Sobolev norm will be denoted by $\|\cdot\|_s$ and if Ω is a domain with boundary we write $\|\cdot\|_{s,\partial}$ for the Sobolev norm on $\partial\Omega$. In particular, the L^2 norm will be denoted by $\|\cdot\|_0$. Sometimes we will be forced to work with fractional order Sobolev spaces spaces whose norm we recall:

$$\|u\|_s = \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^4} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi \right)^{\frac{1}{2}},$$

where $\hat{u}$ is the Fourier transform of u. Fractional Sobolev spaces on domains, manifolds, etc. can be defined with help of a partition of unity.

Repeated indices will be summed. In relativistic problems, Greek indices range from 0 to n and Latin indices from 1 to n, where n is the number of space dimensions. Coordinates are written $(x^0, x^1, \cdots, x^n) = (t, x^1, \cdots, x^n)$ and we write $\partial_t = \partial_0$. In classical problems indices range from 1 to n and are denoted by Latin indices with exception of the compressible free-boundary Euler equations where we use Greek indices ranging from 1 to n and Latin indices from 1 to $n-1$.

If α is a multi index, $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, then D^α denotes the partial derivative of order $|\alpha| = \alpha_0 + \cdots + \alpha_n$ given by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial(x^0)^{\alpha_1}\partial(x^1)^{\alpha_0}\cdots\partial(x^n)^{\alpha_0}}$$

In classical problems, multi-indices always have $\alpha_0 = 0$.

We use both D or ∇ to denote the derivative of a map (a function, n vector field, etc.) and $\partial^k u$ symbolically denotes the k^{th} order derivatives of u (many times, we are interested only in the number of derivatives appearing in the same expression). When dealing with classical(non-relativistic) problems, D and ∇ always denote spatial derivatives; ∂^k represents both space and time derivatives.

We will adopt the philosophy that our quantities are always smooth, even though we are typically interested in a finite number of derivatives. AS it is customary in the field, we obtain results that depend only on say, $\|\cdot\|_s$ norms, and then use a limiting procedure. This allows us to derive estimates in a more direct way (See Jared's lectures as well). We included in these notes some arguments/decisions/calculations that are likely to be omitted from the lectures for the sake of time. These parts of the text are written in gray?

1.2. **References.** We made no attempt in providing complete references or a literature review. In fact, our references are rather incomplete and many important, or even fundamental works are not cited. We only cite references when it directly complements something we say, e.g., a reference for an inequality that we used but did not prove or to a term that we did not define. An exception will occur in the discussion of relativistic viscous fluids because, as it will be seen, a review of the literature is important to set up the problem.

2. The incompressible Euler equations

The Incompressible Euler equations(IEE) describe the motion of a n inviscid incompressible fluid. Inviscid means that the fluid has no viscosity, the latter being the degree to which a fluid under shear sticks to itself. Intuitively, one can think of viscosity as the "stickiness" of the fluid (e.g. honey: high viscosity, water: low viscosity). Incompressibility means that the fluid is volume preserving so it cannot expand or contract (e.g. water at "standard conditions" is well modeled as incompressible, whereas air is always compressible. Needless to say the properties of being inviscid and incompressible are idealizations. (The classical equations describing a fluid with viscosity are the Navier-Stokes equations.)

The IEE are

$$\partial_t v + \nabla_v v + \nabla p = 0 \quad \text{in} \quad [0,T] \times \Omega, \quad \text{(IEEa)}$$
$$\operatorname{div}(v) = 0 \quad \text{in} \quad [0,T] \times \Omega, \quad \text{(IEEb)}$$
$$v \cdot \nu = 0 \quad \text{on} \quad [0,T] \times \partial\Omega, \quad \text{(IEEc)}$$

with initial conditions

$$v(0,\cdot) = v_0 \quad \text{in} \quad \Omega. \quad \text{(IEEd)}$$

The notation is as follows. $\Omega \subseteq \mathbb{R}^4$ is a domain in $\mathbb{R}^4$ (possibly $\Omega = \mathbb{R}^4$). When $\partial\Omega \neq \varnothing$ we will assume it is to be smooth for simplicity. $v = v(t,x) : [0,T] \times \Omega \to \mathbb{R}^4$ is the fluid's velocity. $p = p(t,x) : [0,T] \times \Omega \to \mathbb{R}$ is the fluid's pressure. ∇_v is the (spatial) derivative in the direction of v; componentwise $(\nabla_v z)^i = v^j \frac{\partial z^i}{\partial x^j}$, z a vector field in Ω ($\nabla_v v$ is often written as $(v \cdot \nabla)v$). ∇ is the gradient in $\mathbb{R}^4$. div is the divergence in $\mathbb{R}^4$. ν is the unit outer normal to $\partial\Omega$ and $\cdot$ is the Euclidean inner product.v_0 is a given (divergence free in light of (IEEb) and tangent to $\partial\Omega$ by (IEEc)) vector field in Ω (We will also use ∇_v to denote the directional derivative of a function.)

From the point of view of the initial value problem, the unknown in (IEE) is the velocity v. The pressure p is not an unknown (note that there is not initial condition for p). The pressure is determined from the velocity as follows. Taking divergence of (IEEa) and using (IEEb) gives $\triangle p = -\operatorname{div}(\nabla_v v)$. Restricting (IEEa) to the boundary, taking the inner product with ν and using (IEEc) produces $\frac{\partial p}{\partial \nu} = -\nabla_v v \cdot \nu$, so p satisfies the Newmann problem:

$$\triangle p = -\operatorname{div}(\nabla_v v) \quad \text{in} \quad \Omega$$
$$\frac{\partial p}{\partial \nu} = -\nabla_v v \cdot \nu \quad \text{on} \quad \partial\Omega$$

Writing $p = -\triangle_\nu^{-1}(\nabla_v v)$ to indicate a solution to this boundary value problem (a solution defined up to a constant) we have that ∇p is well-defined. Thus the IEE equations can be written as

$$\partial_t v + \nabla_v v - \nabla \triangle_\nu^{-1}(\nabla_v v) = 0,$$
$$v(0,\cdot) = v_0,$$

and we see that the pressure has been eliminated. Note that the first equation implies that $\operatorname{div}(v)$ is preserved by the time evolution. We see that the IEE are non-local.

Remark 2.1. Any (sufficiently regular) vector field in Ω can be decomposed as a (L^2-orthogonal) sum of a gradient plus a divergence-free and tangent to $\partial\Omega$ part. The operator $\nabla \triangle_\nu^{-1}$ is the projection onto the gradient part. See Leray-Helmholtz projection.

2.1. **Some generalities.** Physically, equation (IEEa) corresponds to Newton's law, i.e., conservation of momentum. (It is possible to add an external force to (IEEa)). Equation (IEEb) is the incompressibility condition. To see this, let $\eta = \eta(t,x)$ be the flow of v, so it satisfies, for each fixed $x \in \Omega$, the ODE $\partial_t \eta(t,x) = v(t,\eta(t,x))$. Let $J(t,x)$ be the Jacobian of the map $x \mapsto \eta(t,x)$. If a fluid is incompressible then $J(t,x) = 1$. But

$$\partial_t J(t,x) = J(t,x)(\mathrm{div} v)(t,\eta(t,x)),$$

(see [37] appendix 1.1 or [36] section 1.3), justifying the claim.

Remark 2.2. The interpretation of $\mathrm{div}(v) = 0$ as incompressibility can also be seen from the formula

$$\mathscr{L}_v(\mathrm{vol}) = \mathrm{div}(v)\,\mathrm{vol}$$

where $\mathscr{L}_v$ is the Lie derivative in the direction of v and vol is a volume form (see [43] chapter 2). (This formula is a particular case of $\mathscr{L}_v w = di_z w + i_z dw$.)

The IEE can be derived directly from Newton's laws (see [37] section 1.1), or form a variational principle (see [?Na]). Regarding the latter, the Lagrangian is

$$L = \frac{1}{2}\int\limits_{\Omega} |v|^2,$$

which also corresponds to the kinetic and total energy of the fluid, a quantity that is conserved (there is no potential energy associated with the IEE). There are otehr conserved quantitties associated with the IEE (see [36] section 1.7) as well as a large set of symmetries (see [36] section 1.2).

We are considering the IEE in $\Omega \subseteq \mathbb{R}^4$ for simplicity, but they can be formulated in a Riemannian manifold (∇ will then be the covariant derivative and the other operators in (IEE) are interpreted in the context of Riemannian geometry; see [44] chapter 17).

2.2. **Local existence and uniqueness.** We will now address the basic question of existence and uniqueness, starting with the latter.

Theorem 2.3. *Let v and u be two smooth solutions to the IEE and defined on the time interval [0,T]. Then:*

$$\sup_{0\le t\le T} \|v(t) - u(t)\|_0 \;\le\; \|v(0) - u(0)\|_0 \exp \int\limits_0^T \|\nabla v(t)\|_{L^\infty} dt$$

In particular, $v = u$ if $v(0) = u(0)$.

Proof. Let $z = v - u$. Then

$$\begin{aligned}
&\partial_t v + \nabla_v v + \nabla p_v = 0\\
&\partial_t u + \nabla_u u + \nabla p_u = 0\\
&\partial_t (v-u) + \nabla_v (v-u) - \nabla_{u-v} u + \nabla(p_v - p_u) = 0
\end{aligned}$$

$$\begin{aligned}
\partial_t z + \nabla_v z + \nabla_z u + \nabla(p_v - p_u) &= 0 \quad \text{in} \quad [0,T]\times\Omega,\\
\mathrm{div}(z) &= 0 \quad \text{in} \quad [0,T]\times\Omega,\\
z\cdot\nu &= 0 \quad \text{on} \quad [0,T]\times\partial\Omega,
\end{aligned}$$

where p_v and p_u are the pressures associated with v and u respectively.

Taking the inner product with z and integrating over Ω:

$$\frac{1}{2}\partial_t \int_\Omega |z|^2 + \int_\Omega z \cdot \nabla_v z + \int_\Omega z \cdot \nabla_z u + \int_\Omega z \cdot \nabla(p_v - p_u) = 0.$$

Integrating by parts (equivalently, using the divergence theorem):

$$\begin{aligned}
&\int_\Omega z \cdot \nabla_v z = -\int_\Omega \nabla_v z \cdot z - \int_\Omega |z|^2 \underbrace{\operatorname{div}(v)}_{=0} + \int_{\partial\Omega} |z|^2 \underbrace{v \cdot \nu}_{=0}, \\
&\Rightarrow \int_\Omega z \cdot \nabla_v z = 0, \\
&\int_\Omega z \cdot \nabla(p_v - p_u) = -\int_\Omega \underbrace{\operatorname{div}(z)}_{=0}(p_v - p_u) + \int_{\partial\Omega} z \cdot \nu(p_u - p_v), \\
&\left| \int_\Omega z \cdot \nabla_z u \right| \leq \|\nabla u\|_{L^\infty} \int_\Omega |z|^2 = \|\nabla u\|_{L^\infty} \|z\|_0^2.
\end{aligned}$$

Writing $\frac{1}{2}\partial_t \int_\Omega |z|^2 = \frac{1}{2}\partial_t \|z\|_0^2 = \|z\|_0 \partial_t \|z\|_0$, dividing by $\|z\|_0$ (which we can assume $\neq 0$), and integrating in time.

$$\|z(t)\|_0 - \|z(0)\|_0 \leq \int_0^t \|\nabla u\|_{L^\infty} \|z\|_0,$$

so that $\|z(t)\|_0 \leq \|z(0)\|_0 \exp\left(\int_0^t \|\nabla u\|_{L^\infty}\right)$ by Grönwall's inequality. The result follows from the fact that $t \in [0, T]$ arbitrary. $\square$

Theorem 2.4. *Let $v_0 \in H^s(\Omega)$, $s > \frac{n}{2} + 1$, be a divergence-free vector field in Ω. Then there exists a $T_* > 0$, depending only on $\|v_0\|_s$ and a*

$$v \in C^0([0, T_*], H^s(\Omega)) \cap C^1([0, T_*], H^{s-1}(\Omega))$$

satisfying the IEE and taking the initial data v_0.

Sketch of Proof The proof follows a similar logic to what is done for quasi-linear wave equation, relying on a combination of a priori estimates and the construction of approximating solutions (to a linearized problem). (Recall Jared's lectures) Her we will restrict ourselves to establishing a priori estimates for smooth solutions.

Very roughly, the idea is as follows. Suppose we want to solve the following initial value problem for a quasi-linear wave equation.

$$\begin{aligned}
g^{\mu\nu}(u, \partial u)\partial_\mu \partial_\nu u &= f(u, \partial u), \\
u(0, \cdot) &= \mathring{u}_0, \\
\partial_t u(0, \cdot) &= \mathring{u}_1,
\end{aligned}$$

where $g^{\mu\nu}(u, \partial u)$ indicates that g is a Lorentzian metric that is a function of u and first derivatives of u; $f(u, \partial u)$ indicates that the RHS is a function of u and the first derivatives of u. $\mathring{u}_0$ and $\mathring{u}_0$, are given initial conditions (belonging to some appropriate function space). μ, ν vary from 0 to n with $\partial_0 = \partial_t$.

The equation is solved as follows. Define a sequence $\{u_\ell\}$ inductively upon solving the linear problem (which is treated by standard linear theory):

$$\begin{aligned} g^{\mu\nu}(u_\ell, \partial u_\ell)\partial_\mu\partial_\nu u_{\ell+1} &= f(u_\ell, \partial u_\ell), \\ u_{\ell+1}(0,\cdot) &= \mathring{u}_0, \\ \partial_t u_\ell(0,\cdot) &= \mathring{u}_1, \end{aligned}$$

with $u_0 = \mathring{u}_0$. For each ℓ we have an energy estimate for the $u_{\ell+1}$. Using the energy estimate we can show that if we restrict $u_{\ell+1}$ to a sufficiently small time then the sequence $\{u_\ell\}$ converges (in some appropriate function space) to a limit u_∞ (that is why the solution to quasi-linear problems is guaranteed to exist only on a small time interval). From the equation we see that u_∞ solves the quasi-linear equation. The crucial part in this argument is the use of energy estimates to ensure convergence. (See [41] chapter 9 for details.) That is one reason to study a priori estimates. Precisely the same logic (constructing a sequence from linear problem etc.) applies to fluids. What is very different is the a priori estimates. In fact, this philosophy is applicable to many evolution equations, hence the importance of studying a priori estimates.

Apply D^α to (IEEa) where $\alpha =$ multi-index, take the inner product with $D^\alpha v$ and integrate over Ω and sum over $|\alpha| \le s$

$$\frac{1}{2}\sum_{|\alpha|\le s}\partial_t\int_\Omega |D^\alpha v|^2 + \sum_{|\alpha|\le s}\int_\Omega D^\alpha v\cdot D^\alpha(\nabla_v v) + \sum_{|\alpha|\le s}\int_\Omega D^\alpha v\cdot D^\alpha \nabla p = 0$$

We have, using Cauchy-Schwartz and integrating by parts:

$$\begin{aligned} \left|\sum_{|\alpha|\le s}\int_\Omega D^\alpha v\cdot D^\alpha(\nabla_v v)\right| &\le \sum_{|\alpha|\le s}\int_\Omega |D^\alpha v\cdot(D^\alpha(\nabla_v v) - \nabla_v(D^\alpha v))| + \sum_{|\alpha|\le s}\left|\int_\Omega D^\alpha v\cdot\nabla_v D^\alpha v\right| \\ &\le \sum_{|\alpha|\le s}\int_\Omega |D^\alpha v\cdot(D^\alpha(\nabla_v v) - \nabla_v(D^\alpha v))| \end{aligned}$$

where we used:

$$\begin{aligned} &\int_\Omega D^\alpha v\cdot\nabla_v D^\alpha v = -\int_\Omega \nabla_v D^\alpha v\cdot D^\alpha v - \int_\Omega |D^\alpha v|^2\underbrace{\operatorname{div}(v)}_{=0} + \int_\Omega |D^\alpha v|^2\underbrace{v\cdot\nu}_{=0} \\ &\Rightarrow \int_\Omega D^\alpha v\cdot\nabla_v D^\alpha v = 0, \\ &\left|\sum_{|\alpha|\le s}\int_\Omega D^\alpha v\cdot D^\alpha(\nabla_v v)\right| \le \sum_{|\alpha|\le s}\|D^\alpha v\|_0\|D^\alpha(\nabla_v v) - \nabla_v(D^\alpha v)\|_0 \\ &\le C\|v\|_s\sum_{|\alpha|\le s}(\|\nabla v\|_{L_\infty}\|D^s v\|_0 + \|D^s v\|_0\|\nabla v\|_{L^\infty}) \\ &\le C\|v\|_s^2\|\nabla v\|_{L^\infty} \end{aligned}$$

where we used (Moser's inequality, [36] ch. 3,[44] ch.17)

$$\sum_{|\alpha|\leq s}\|D^\alpha(fg)-fD^\alpha g\|_0\leq C(\|\nabla f\|_{L^\infty}\|D^{r-1}g\|_0+\|D^r f\|_0\|g\|_{L^\infty})$$

(with $g\mapsto\nabla v$; we also used $D^\alpha\nabla=\nabla D^\alpha$.)

To estimate the term with ∇p recall that p satisfies

$$\begin{aligned}\triangle p &= -\mathrm{div}(\nabla_v v)\quad\text{in}\quad\Omega\\ \frac{\partial p}{\partial\nu}&=-\nabla_v v\cdot\nu\quad\text{on}\quad\partial\Omega\end{aligned}$$

But $\mathrm{div}(\nabla_v v)=\partial_i v^j\partial_j v^i$ (by (IEEb) and since ∇_v is tangential to $\partial\Omega$, $v\cdot\nu=0\Rightarrow\nabla_v v\cdot\nu=-v\cdot\nabla_v\nu$. Thus

$$\begin{aligned}\triangle p &= \partial_i v^j\partial_j v^i\quad\text{in}\quad\Omega,\\ \frac{\partial p}{\partial\nu}&=v\cdot\nabla v\nu\quad\text{on}\quad\partial\Omega.\end{aligned}$$

Elliptic theory now gives, for $r\geq 0$,

$$\begin{aligned}\|p\|_{r+2}&\leq C(\|\partial_i v^j\partial_j v^i\|_r+\|v\cdot\nabla_v\nu\|_{r+\frac{1}{2},\partial})\\ &\leq C(\|D^{r+1}v\|_0\|\nabla v\|_{L^\infty}+\|v\cdot\nabla_v\tilde{\nu}\|_{r+1})\\ &\leq C(\|D^{r+1}v\|_0\|\nabla v\|_{L^\infty}+\|D^{r+1}v\|_0\|\nabla_v\tilde{\nu}\|_{L^\infty}+\|v\|_{L^\infty}\|\nabla_v\tilde{\nu}\|_{r+1})\\ &\leq C\|v\|_{C^1}\|D^{r+1}v\|_0,\end{aligned}$$

where we used the inequality (Moser's inequality, same references as above)

$$\|fg\|_r\leq C(\|D^r f\|_0\|g\|_{L^\infty}+\|f\|_{L^\infty}\|D^r g\|_0),$$

the restriction inequality $\|f\|_{r-\frac{1}{2},\partial}\leq C\|f\|_r$, $r>\frac{1}{2}$ (See [39], chapter X), and we extend ν to a smooth vector field $\tilde{\nu}$ in Ω (this can be made by extending ν to a neighborhood of $\partial\Omega$ with the help of a partion of unity and using a bump function to make it zero away from $\partial\Omega$). Note that the norms of $\tilde{\nu}$ are absorbed into C. Then

$$\begin{aligned}\sum_{|\alpha|\leq s}\int_\Omega D^\alpha v\cdot D^\alpha\nabla p&\leq\sum_{|\alpha|\leq s}\|D^\alpha v\|_0\|D^\alpha\nabla p\|_0\leq C\|v\|_s\|p\|_{s+1}\\ &\leq C\|v\|_s^2\|v\|_{C^1},\end{aligned}$$

using the estimate for p with $r=s-1$.

Combining the estimates gives $\partial_t\|v\|_s^2\leq C\|v\|_{C^1}\|v\|_s^2$. Using Grönwall's inequality after integrating in time:

$$\|v\|_s\leq\|v(0)\|_s\exp\left(C\int_0^t\|v\|_{C^1}\right)$$

which is the basic a priori estimate that can be used, as in the case of quasi-linear wave equations to construct solutions. □

Let's make some remarks about the proof.

If $\Omega = \mathbb{R}^4$ (or Π^4), then integration by parts gives

$$\int_{\Omega} D^\alpha v \cdot D^\alpha \nabla p = -\int_{\Omega} D^\alpha \mathrm{div}(v) D^\alpha p = 0$$

by (IEEb). The above proof, with Ω a domain with boundary, illustrates how the introduction of boundaries carries difficulties (as will be the case for free-boundary problems).

The assumption $s > \frac{n}{2} + 1$ is needed because when we construct solutions, we have to bound(as in the case of quasi-linear wave equations) $\|v\|_{C^1}$ by $\|v\|_s$, which is done using the Sobolev embedding theorem: $\|v\|_{C^1} \leq C\|v\|_s$ if $s > \frac{n}{2} + 1$.

We can obtain a rough estimate for the time existence as follows. From the inequality $\partial_t \|v\|^2 \leq C\|v\|_{C^1}\|v\|_s^2$ and Sobolev embedding we have $\partial_t \|v\| \lesssim \|v\|_s^2$. Thus if $\|v\|_s$ remains finite up to time T: $\int_0^T \frac{d\|v\|_s}{\|v\|_s^2} \lesssim \int_0^T dt$, so that $-\frac{1}{\|v\|_s} + \frac{1}{\|v(0)\|_s} \lesssim T$, or $\|v\|_s \lesssim \frac{\|v(0)\|_s}{1 - T\|v_0\|_s}$. $\|v\|_s$ remains finite as $1 \cdot T\|v_0\|_s$, which gives the rough estimate for time existence: $T \ \frac{1}{\|v_0\|_s}$.

We finally turn to the question of continuation of solutions versus blow-up. Once again as in the case of quasi-linear wave equations, we can use the estimate:

$$\|v\|_s \leq \|v(0)\|_s \exp\left(C \int_0^T \|v\|_{C^1} \right)$$

to show that if $\limsup_{t \uparrow T} \|v(t)\|_{C^1} < \infty$ then the solution can be continued (in H^s) past T. What is remarkable in the case of the IEE (and does not have an analogue in quasi-linear wave equations) is the famous Beale-Kato-Majda (BKM) criterion, which states that $\|v\|_{C^1}$ can be controlled by $\|\omega\|_{L^\infty}$ where ω is the vorticity of the fluid, defined as (for $n = 2$ or 3),

$$\omega = \mathrm{curl}(v)$$

or, in components, $\omega^i = \varepsilon^{ijk}\partial_j v_k$ where ε^{ijk} is the totally anit-symmetric symbol (Levi-Civita symbol). (For $n = 2$, we think of v as a vector field $(v, 0) \in \mathbb{R}^3$; then ω is orthogonal to the x^1-x^2-plane and can be identified with a function of $\mathbb{R}^2$).

(See [36] chapter 3 or [44] chapter 17 for a precise statement of the BKM criterion.)

The BKM criterion is interesting because (a) it ties the problem of global existence vs. blow-up to the vorticity, which is a quantity with physical meaning and extremely relevant for the study of turbulence (see [36] for an introduction to to the mathematics of turbulence); and (b) the vorticity satisfies a transport like equation that can be used to study it. In particular using such an equation we can show that $\omega(t, x) = \omega(0, \eta(-t, x))$, where η is the flow of v. From this it follows that solutions to the Euler equations exist globally when $n = 2$ (see [36] for details). Global existence or blow up for the IEE in $n = 3$ is one of the big open problems in mathematical fluid dynamics.

Remark 2.5. The fact that ω controls v can be seen from the estimate

$$\|X\|_s \leq C(\|\mathrm{div}(X)\|_{s-1} + \|\mathrm{curl}(X)\|_{s-1} + \|X \cdot \nu\|_{s-\frac{1}{2},\partial} + \|X\|_0)$$

valid for any (sufficiently smooth) vector field X (this estimate is well known; see [5] for a modern proof). Since $\mathrm{div}(v) = 0$, $v \cdot \nu = 0$, and $\|v\|_0$ is conserved for v a solution to the IEE, only $\mathrm{curl}(v)$ matters. However, here we need control of $\mathrm{curl}(v)$ in H^{s-1} (which is as hard as controlling v directly in H^s), whereas in the BKM criterion we only need to control $\mathrm{curl}(v)$ in L^∞

2.3. **Equation for the vorticity.** Here we derive the equation for the vorticity mentioned earlier. Taking the curl of (IEEa):

$$\underbrace{\mathrm{curl}(\partial_t v)}_{=\partial_t \mathrm{curl}(v)=\partial_t\omega} + \mathrm{curl}(\nabla_v v) + \underbrace{\mathrm{curl}(\nabla p)}_{=0 \quad \text{since} \quad \mathrm{curl}\nabla=0} = 0$$

Compute

$$\begin{aligned}(\mathrm{curl}(\nabla_v v))^i &= \varepsilon^{ijk}\partial_j(\nabla_v v)_k = \varepsilon^{ijk}\partial_j(v^\ell\partial_\ell v_k)\\ &= \underbrace{v^\ell\partial_\ell(\underbrace{\varepsilon^{ijk}\partial_j v_k}_{\omega^i})}_{=(\nabla_v\omega)^i} + \varepsilon^{ijk}\partial_j v^\ell\partial_\ell v_k\end{aligned}$$

Thus

$$\underbrace{\partial_t\omega^i + (\nabla_v\omega)^i}_{\text{transport operator (along } v\text{) applied to }\omega.} + \varepsilon^{ijk}\partial_j v^\ell\partial_\ell v_k = 0$$

Note the similarity of the first two terms with the first two terms of the IEE.

In two dimensions (and considering $(v,0)\in\mathbb{R}^3$), the term $\varepsilon^{ijk}\partial_j v^\ell\partial_\ell v_k$ vanishes. For $\partial_j v^\ell\partial_\ell v_k = 0$ whenever j or $k=3$. But we also have:

$$\begin{aligned}\varepsilon^{ijk}\partial_j v^\ell\partial_\ell v_k &= \partial_1 v^\ell\partial_\ell v_2 - \partial_2 v^\ell\partial_\ell v_1\\ &= \partial_1 v^1\partial_1 v_2 + \partial_1 v^2\partial_2 v_2 - \partial_2 v^1\partial_1 v_1 - \partial_2 v^2\partial_2 v_1\\ &= \underbrace{(\partial_1 v^1 + \partial_2 v^2)}_{=0}\partial_1 v^2 - (\underbrace{\partial_1 v^1+\partial_2 v^2}_{=0})\partial_2 v^1 = 0.\end{aligned}$$

Also only ω^3 is non-zero for $n=2$. So in two dimensions the vorticity satisfies a transport equation:

$$\partial_t\omega + \nabla_v\omega = 0$$

giving $\omega(t,\eta(t,y)) = \omega(0,y)$ or $\omega(t,x) = \omega(0,\eta(-t,x))$, since the inverse of the map $y\mapsto \eta(t,y)$ is $x=\eta(t,y)\mapsto y-\eta(-t,x)$.

It is useful to write the equation for the vorticity in a more geometric fashion. Consider

$$\frac{1}{2}(\nabla|v|^2)^i = \frac{1}{2}\partial^i(v^\ell v_\ell) = v^\ell\partial^i v_\ell = \overbrace{v^\ell\partial_\ell v^i}^{=(\nabla_v v)^i} + (v^\ell\partial^i v_\ell - v^\ell\partial_\ell v^i).$$

Compute

$$(v\times\omega)^i = \varepsilon^{ijk}v_j\varepsilon_k{}^{\ell n}\partial_\ell v_n$$

But $\varepsilon^{ijk}\varepsilon_{k\ell n} = \varepsilon^{kij}\varepsilon_{k\ell n} = \delta^i_\ell\delta^j_n - \delta^j_\ell\delta^i_n$, so $(v\times\omega)^i = (\delta^{i\ell}\delta^{jn}-\delta^{j\ell}\delta^{in})v_j\partial_\ell v_n = v^n\partial^i v_n - v^\ell\partial_\ell v^i$.

We conclude the identity $\nabla_v v = \frac{1}{2}\nabla|v|^2 - v\times\omega$.

Taking the curl:

$$\begin{aligned}\mathrm{curl}(\nabla_v v) &= \mathrm{curl}(v\times\omega). \text{ Computing}\\ (\mathrm{curl}(v\times\omega))^i &= \varepsilon^{ijk}\partial_j(v\times\omega)_k = \varepsilon^{ijk}\partial_j(\varepsilon_k{}^{ln}v_\ell\omega_n)\\ &= \varepsilon^{ijk}\varepsilon_k{}^{ln}\partial_j v_\ell\omega_n + \varepsilon^{ijk}\varepsilon_k{}^{ln}v_\ell\partial_j\omega_n\\ &= (\delta^{i\ell}\delta^{jn}-\delta^{j\ell}\delta^{in})\partial_j v_\ell\omega_n + (\delta^{i\ell}\delta^{jn}-\delta^{j\ell}\delta^{in})v_\ell\partial_j\omega_n\\ &= \partial^n v^i\omega_n - \underbrace{\partial_\ell v^\ell}_{=0}\omega^i + v^i\underbrace{\partial_n\omega^n}_{=0} - \omega^j\partial_j\omega^i\end{aligned}$$

$$= (\nabla_\omega v)^i - (\nabla_v \omega)^i.$$

Therefore:

$$\partial_t \omega - \nabla_v \omega - \nabla_\omega v = 0.$$

The precise estimate for v in terms of ω is

$$\|v\|_{C^1} \leq C((1 + \log^+(\|v\|_s))\|\omega\|_{L^\infty} + 1)$$

see [21], where

$$log^+ x = \begin{cases} \log x, & x > 1 \\ 0, & x \leq 1. \end{cases}$$

The $\|v\|_s$ term the above estimate is not a problem because in the energy estimate $\|v\|_{C^1}$ appears inside a time integral (so we can use Grönwall-like arguments).

Remark 2.6. $\|v\|_{C^1}$ can in fact be replaced by $\|v\|_{W^{1,\infty}}$.

3. The compressible Euler equations

For the IEE, the density of the fluid was constant (since the fluid could not contract or expand) and was, therefore, conviniently set to one in euqations (IEE). If the fluid density is allowed to change, then we have the compressible Euler Equations (CEE):

$$\partial_t v + \nabla_v v + \frac{1}{\varrho}\nabla p = 0 \quad \text{in} \quad [0,T] \times \Omega \quad \text{(CEEa)},$$
$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \quad \text{in} \quad [0,T] \times \Omega \quad \text{(CEEb)},$$
$$p = p(\varrho) \quad \text{in} \quad [0,T] \times \Omega \quad \text{(CEEc)},$$
$$v \cdot \nu = 0 \quad \text{on} \quad [0,T] \times \partial\Omega \quad \text{(CEEd)},$$

with initial conditions

$$v(0,\cdot) = v_0 \quad \text{in} \quad \Omega \quad \text{(CEEe)}$$
$$\varrho(0,\cdot) = \varrho_0 \quad \text{in} \quad \Omega \quad \text{(CEEf)}$$

Compared to the IEE, the new element now is the density of the fluid, $\varrho = \varrho(t,x) : [0,T] \times \Omega \leftarrow \mathbb{R}_+$ (physically, the density has to be positive; we will discuss the possibility $\varrho = 0$ when we study free boundary problems). Another important difference is that now the pressure is not determined by v, but rather by the equation (CEEc), known as equation of state: this is a given relation between the pressure and the density whose nature depends on the nature of the fluid (e.g., $p(\varrho) = A\varrho^\mu + B$ with A, B and p constants that are typically determined experimentally).

From the point of view of the initial value problem, the unknowns are v and ϱ. (Alternative, using that $p = p(\varrho)$ is invertible for physical equations of state, we can take v and p as unknowns and determine ϱ by $\varrho = \varrho(p)$.)

Remark 3.1. In view of (CEEa)-(CEEd), the initial conditions v_0 and ϱ_0 cannot be arbitrary but we need to satisfy compatibility conditions. (Note, also, that unlike the IEE, here v_0 need not be divergence-free) (As an analogy, say we want to solve $-u_{tt} + u_{xx} = 0$ in $(0,\infty) \times [0,1]$, $u(0,x) = g(x), \partial_t u(0,x) = h(x)$, with boundary conditions $u(t,0) = 0$, $u(t,1) = 0$. Then g and h have to satisfy the compatibility conditions $g(0) = g(1) = 0$, $h(0) = h(1) = 0$.)

Remark 3.2. Equations (CEE) are sometimes called the isentropic compressible Euler equations, isentropic meaning that entropy is not included in the equations.

We need to make reasonable (compatible with physics) assumptions about the equations of state. We will assume that $p : (0,\infty) \to (0,\infty)$ is a 1-1, smooth, strictly increasing function. (See [35] for a discussion.)

3.1. **Compatibility conditions.** Typically, when solving the CEE, we look for solutions with a finite number of derivatives (say, in H^s). When the restriction of derivatives $D^k v$ and $D^k \varrho$ of v and ϱ to $\partial\Omega$ is well-defined, equations (CEE) impose relations between v and ϱ. Such relations have to hold, in particular, at $t = 0$, thus for v_0 and ρ_0.

The zeroth order compatibility condition is simply the boundary condition: $v_0 \cdot \nu = 0$ on $\partial\Omega$.

Differentiating (CEEd) and setting $t = 0$ thus $\partial_t v \cdot \nu|_{t=0} = 0$, so that evaluating (CEEa) at $t = 0$, restricting it to $\partial\Omega$ and dotting with ν gives:

$$(\partial_{v_0} v_0 + \frac{p'(\rho_0)}{\rho_0} \nabla \rho_0) \cdot \nu = 0,$$

which is the first order compatibility condition.

Differentiating (CEEa) with respect to t:

$$\partial_t^2 v + \nabla_{\partial_t v} v + \nabla_v \partial_t v + \partial_t \left(\frac{p'(\rho)}{\rho} \right) \nabla \rho + \frac{1}{\rho} \nabla \partial_t \rho = 0$$

$$\partial_t^2 v + \nabla_{\partial_t v} v + \nabla_v \partial_t v + \left(\frac{p'(\rho)}{\rho} \right)' \partial_t \nabla \rho + \frac{1}{\rho} \nabla \partial_t \rho = 0$$

Using (CEEb):

$$\partial_t^2 v + \nabla_{\partial_t v} v + \nabla_v \partial_t v - \left(\frac{p'(\rho)}{\rho} \right)' \operatorname{div}(\rho v) \nabla \rho + \frac{1}{\rho} \nabla \operatorname{div}(\rho v) = 0$$

We will restrict this expression to $\partial\Omega$ and dot it with ν. Note that $\partial_t^2 v \cdot \nu = 0$. Introducing the second fundamental form of $\partial\Omega$:

$$k(X, Y) = \nabla_X Y \cdot \nu = -X \cdot \nabla_Y \nu,$$

for X, Y tangent to $\partial\Omega$.

Then (using that k is symmetric)

$$\nabla_{\partial_t v} v \cdot \nu + \nabla_v \partial_t v \cdot \nu = 2k(v, \partial_t v),$$

and we obtain as the second order compatibility condition:

$$-2k \left(n_0, \nabla_{v_0} v_0 + \frac{1}{\rho_0} p'(\rho_0) \nabla \rho_0 \right) - \left(\frac{p'(\rho)}{\rho} \right)' \Big|_{\rho_0} \operatorname{div}(\rho_0 v_0) \frac{\partial \rho}{\partial \nu} - \frac{1}{\rho_0} \frac{\partial}{\partial \nu} (\operatorname{div}(\rho_0 v_0)) = 0.$$

We can continue and derive higher order compatibility conditions. The ℓ^{th} order compatibility condition will involve up to ℓ derivatives of ρ_0 and up to $\ell - 1$ derivatives of v_0 and $\operatorname{div}(v_0)$.

To obtain solutions in H^s, we need (v_0, ρ_0) to satisfy the compatibility conditions up to order $s - 1$.

3.2. **Local existence and uniqueness.** We now investigate local existence and uniqueness for (CEE):

Theorem 3.3. *Let $v_0 \in H^s(\Omega)$, $\rho_0 \in H^s(\Omega)$, $s > \frac{n}{2} + 1$. Assume that v_0 and ρ_0, satisfy the compatibility conditions up to order $s-1$. Suppose that Ω is bounded and that $\rho \leq constant > 0$. Let an equation of state be given with the properties previously stated. Finally, assume that $|v_0(x)|^2 < p'(\rho)(x)$ for all $x \in \Omega$.*

Then, there exists a $T_* > 0$, depending only on $\|v_0\|_s$ and $\|\rho_0\|_s$, and unique

$$v \in C^0([0,T_*], H^s(\Omega)) \cap C^1([0,T_*], H^{s-1}(\Omega)),$$
$$\rho \in C^0([0,T_*], H^s(\Omega)) \cap C^1([0,T_*], H^{s-1}(\Omega))$$

satisfying the CEE and taking the initial data (v_0, ρ_0).

Sketch of the proof: Rewrite (CEEa)-(CEEb) as

$$\partial_t v + \nabla_v v + \frac{1}{\rho} p'(\rho) \nabla \rho = 0,$$
$$\partial_t \rho + \nabla_v \rho + \rho \operatorname{div}(v) = 0.$$

Multiplying the first equation by ρ and the second by $\frac{p'(\rho)}{\rho}$:

$$\rho \partial_t v + \rho \nabla_v v + p'(\rho) \nabla \rho = 0$$
$$\frac{p'(\rho)}{\rho} \partial_t \rho + p'(\rho) \operatorname{div}(v) + \frac{p'(\rho)}{\rho} \nabla_v \rho = 0$$

so that

$$\begin{pmatrix} \rho & 0 \\ 0 & \frac{p'(\rho)}{\rho} \end{pmatrix} \partial_t \begin{pmatrix} v^i \\ \rho \end{pmatrix} + \begin{pmatrix} \rho \nabla_v & p'(\rho)\partial_i \\ p'(\rho)\partial_i & \frac{p'(\rho)}{\rho}\nabla_v \end{pmatrix} \begin{pmatrix} v^i \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

More explicitly

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \frac{p'(\rho)}{\rho} \end{pmatrix} \partial_t \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ \rho \end{pmatrix} + \begin{pmatrix} \rho v^1 & 0 & 0 & p'(\rho) \\ 0 & \rho v^1 & 0 & 0 \\ 0 & 0 & \rho v^1 & 0 \\ p'(\rho) & 0 & 0 & \frac{p'(\rho)}{\rho} v^1 \end{pmatrix} \partial_1 \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ \rho \end{pmatrix}$$
$$+ \begin{pmatrix} \rho v^2 & 0 & 0 & 0 \\ 0 & \rho v^2 & 0 & p'(\rho) \\ 0 & 0 & \rho v^2 & 0 \\ 0 & p'(\rho) & 0 & \frac{p'(\rho)}{\rho} v^2 \end{pmatrix} \partial_2 \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ \rho \end{pmatrix} + \begin{pmatrix} \rho v^2 & 0 & 0 & 0 \\ 0 & \rho v^3 & 0 & 0 \\ 0 & 0 & \rho v^2 & p'(\rho) \\ 0 & 0 & p'(\rho) & \frac{p'(\rho)}{\rho} v^3 \end{pmatrix} \partial_3 \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Written in this form, the system forms a (quasi-linear) first order symmetric hyperbolic system for which known results can be invoked (under the assumptions of the theorem). □

Let us make some comments. We recall that a first order system of PDE's

$$A^0(u)\partial_0 u + A^i(u)\partial_i u + B(u) = 0$$

is said to be symmetric hyperbolic for a function h of the matrix $A^0(h)$ is positive definite and the matrices $A^0(h)$, and $A^i(h)$ are symmetric.

There are many works addressing existence and uniqueness of quasi-linear symmetric hyperbolic systems in $\mathbb{R}^4$ (see, e.g., [35] or [32]). In the case of domains with boundary, the literature seems to be more restrictive, but a proof of local well-posedness can be found in

[17]. This brings us to the assumption $|v_0|^2 < p'(\rho_0)$. This is a technical assumption that is not needed in $\mathbb{R}^4$, but is used in the case of bounded domains. (In a nutshell, one tries, as usual, to construct a map upon solving the associated linear problem and then show that this map is a contradiction. To do so, we work in a space of functions that have the property of satisfying the compatibility conditions at time zero. The assumption $|v_0|^2 < p'(\rho_0)$ is used to show that such space is not empty. Obviously, this issue does not arise in $\mathbb{R}^4$.) This is another example of how the presence of boundaries can cause difficulties. While it is possible that the need for $|v_0|^2 < p'(\rho_0)$ can be an artifact of the method used, it is interesting to note that it has a clear physical interpretation as follows.

It can be showed (see [35]) that $\sqrt{p'(\rho)}$ corresponds to the sound speed of the fluid, i.e., the speed of propagation of sound waves within the fluid. Thus, $|v_0|^2 < p'(\rho_0)$ says that the fluid's velocity is everywhere less than the fluid's sound speed at $t = 0$ (note that the sound speed is a function of time and space), i.e., the fluid is sub-sonic. (Note that under our assumptions $p'(\rho) > 0$, so $\sqrt{p'(\rho)}$ makes sense.)

3.3. **The incompressible limit.** It is natural to ask how equations (IEE) and (CEE) are related. Since, physically, (IEE) describes a fluid with $\rho = 1$, we can expect that the CEE reduce to the IEE when $\rho = 1$. Formally this is the case, since plugging $\rho = 1$ into (CEE) gives

$$\partial_t v + \nabla_v v + \nabla p = 0 \quad \text{and} \quad \operatorname{div}(v) = 0,$$

which seemingly produces the IEE. This is not quite correct, however. This is a formal calculation that ignores the fact that $p = p(\rho)$. Taking the equation of state into account, we have, setting $\rho = 1$, that p is constant, so that $\nabla p = 0$ and (CEEa) becomes

$$\partial_t v + \nabla_v v = 0$$

which is not (IEEa).

It is legitimate to ask whether there is a sense in which (CEE) reduces to (IEE). It is worth noticing that mathematically these equations are quire different. As seen, (CEE) can be written as a first order symmetric hyperbolic system, thus the CEE enjoy finite speed of propagation. The IEE, on the other hand, are non-local (due to the pressure, as seen), exhibiting infinite propagation speed.

The problem of the relation between equations (CEE) and (IEE) is referred to as the incompresible limit (a.k.a the limit of zero Mach number). Said a bit less vaguely, the incompressible limit consists in showing that solutions to (CEE) converge to a solution to (IEE) when some notion of "compressibility" goes to zero. The correct way of stating this is via the sound speed, i.e., the incompressible limit corresponds to the limit when the sound speed goes to ∞ (so "compressiblity" can be defined as $\frac{1}{\sqrt{p'(\rho)}}$.

See [35] or [13] for a precise definition of the incompressible limit. The incompressible limit in $\mathbb{R}^4$ or Π^4 has been studied by many authors. For a proof in the case of a bounded domain, see [13](see [13] also for a review of the literature).

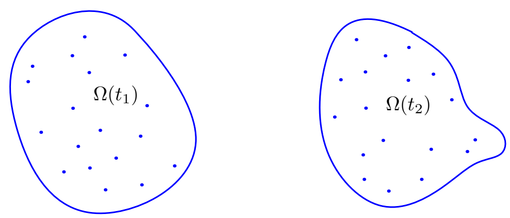
The notion that the incompressible limit corresponds to the sound speed going to ∞ comes from the fact that stiffer fluids have larger sound speed. For example:

Material	Sound speed(ft/s)
Air	1117
Water	4890
Glycerin	6100
Ice	10500
Steel	16600

(Source [46])

4. The free boundary Euler equations

In many situations of interest, the region Ω containing the fluid is not fixed but is allowed to move with the fluid.



fluid at time t_1 fluid at time t_2

In this case, the domain containing the fluid becomes a time dependent object that depends on the fluid motion. Examples of this situation are a liquid drop or star. The equations describing such a scenario are the free-boundary Euler equations.

As in the case of a fixed domain, the free-boundary Euler equations can be considered for compressible or incompressible fluids (both situations are discussed below).

Remark 4.1. Strictly speaking the free boundary Euler equations model a fluid region in vacuum. The situation of, for instance, a water drop in the air, is more correctly described by a two-phase fluid model. In such case we have two fluids (water and air) interact through a common interference that moves with the fluids. However, given that the density of air is much smaller than that of water, we can approximate this situation by the case of a water drop in vacuum and thus employ the free boundary Euler equations. (Note that the realistic situation of a water drop would have to include the force of gravity as well, but we will not do it here.) These simplifications notwithstanding, it should be remarked that many of the ideas we will discuss for free-boundary problems can be adapted to the study of two-phase fluids.

A related problem is the study of a fluid interacting with a "structure" (typically, an elastic body), in such a way that the boundary of the structure moves according to the flow dynamics. An example is blood flowing through an artery (blood=fluid, artery=structure). Problems of this type are known as fluid structure interaction. The free-boundary problems ideas that we will present can also be adapted to fluid structure interaction problems.

Conceptually, the main difficulty in dealing with free-boundary problems comes from the moving domain: we want to solve a system of PDEs, but the very domain where the equations are defined depends on the unknowns (the fluid velocity, etc); i.e., the domain of definition of the PDEs is also one of the unknowns of the problem.

(As we will see, we can reparametrize the moving domain in such a way that the equations can be rewritten in terms of a fixed domain. But this will introduce now non-linearities.)

The study of free-boundary problems has some significant differences compared to the study of fluid equations in a fixed domain or quasi-linear wave equations. For these problems, local existence is established by the traditional method of a priori estimates plus iteration; so we say, loosely speaking, that for such equations "existence follows from a priori estimates."" The situation is radically different for the free-boundary Euler equations. The a priori estimates now depend on very specific features of the equations and are, therefore, very sensitive to perturbations. Consequently, the associated linear problem typically does not provide a good model for constructing approximating solutions.

Furthermore, the a priori estimates themselves are challenging due to the presence of the moving boundary. To close the estimates, we have to exploit the full non-linear structure of the equations as well as underlying geometry.

We will illustrate these prints in two ways.

First, we will outline a proof of local existence and uniqueness of the incompressible free-boundary Euler equations wherein the geometry plays a prominent role. Second, we will sketch a derivation of a priori estimates for the compressible Euler equations, highlighting the special structures involved. We will also illustrate how the traditional way of deriving a priori estimates (roughly, differentiating the equations and applying a L^2- energy inequality) fails for the free boundary Euler equations.

5. The incompressible free-boundary Euler equations

The incompressible free-boundary Euler equations (IFBEE) are:

$$\begin{aligned} \partial_t u + \nabla_u u + \nabla p = 0 \quad &\text{in} \quad D \quad \text{(IFBEEa)},\\ \operatorname{div}(u) = 0 \quad &\text{in} \quad \mathscr{D} \quad \text{(IFBEEb)},\\ p = \sigma \mathscr{H} \quad &\text{on} \quad \partial D \quad \text{(IFBEEc)}\\ \partial_t + u^i \frac{\partial}{\partial x^i} &\in T(\partial D) \quad \text{(IFBEEd)}, \end{aligned}$$

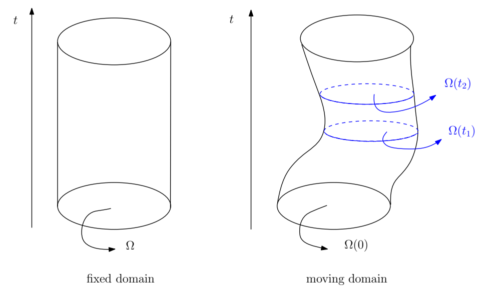
where

$$D = \bigcup_{0 \le t < T} \{t\} \times \Omega(t) \quad \text{(IFBEEe)}$$

with initial conditions

$$\begin{aligned} u(0,\cdot) &= u_0 \quad \text{(IFBEEf)},\\ \Omega(0) &= \Omega_0 \quad \text{(IFBEEg)} \end{aligned}$$

The notation is as follows. $\Omega(t)$ is the moving domain at time t, which has to be determined from the equations (later on we will give a more explicit description of $\Omega(t)$ that makes its dependence on the fluid motion more apparent). The dynamics (the domain of definition of the equation) takes place in D. For comparison, had the equations been defined in a fixed domain Ω we would have $D = [0,T) \times \Omega$. The difference between the two situations is illustrated in the following picture.



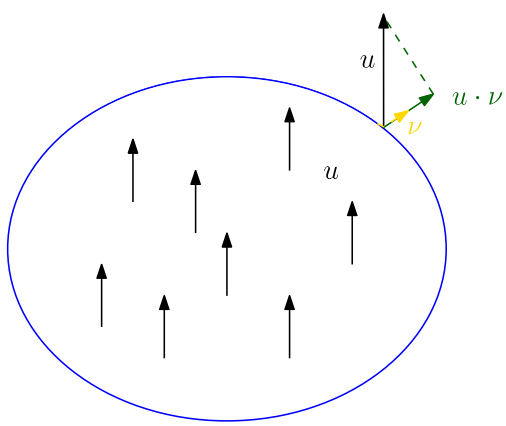
$u = u(t,x) : D \to \mathbb{R}^4$, $p = p(t,x) : D \to \mathbb{R}$, are the velocity and pressure of the fluid; the notation $(t,x) \in D$ being that for each $t \in [0,T)$ we have $x \in \Omega(t)$. σ is a non-negative constant known as coefficient of surface tension. $\mathscr{H}$ is the mean curvature of the embedding of $\partial\Omega(t)$ into $\mathbb{R}^4$. $T(\partial D)$ is the tangent bundle of ∂D. u_0 is a given (divergence-free by (IFBEEb)) vector field in Ω_0, and Ω_0 is a given domain.

From the point of view of the initial value problem, the uknowns are u, p, and D (or, equivalently, $\Omega(t)$.

Remark 5.1. A fundamental difference between the IEE and the IFBEE is that for the latter the pressure is a "honest" unknown.

The quantity $\sigma\mathscr{H}$ is called the surface tension of the fluid. The IFBEE behave very differently depending on whether $\sigma = 0$ or $\sigma > 0$, which we refer to as the IFBEE with or without surface tension. Here we will deal with the case $\sigma > 0$. Thinking of the example of a water drop in air, the surface tension results from the fact that the force of attraction among water molecules, so that the surface tension is responsible for the cohesion of the liquid drop.

Equation (IFBEEd) says that $\partial\Omega(t)$ moves at a speed equal to the normal component of the fluid's velocity.



Remark 5.2. A fluid is called irrotational if $\omega = \operatorname{curl}(u) = 0$. This is a condition that is propagated by the flow (i.e., $\omega(t) = 0$ if $\omega(0) = 0$). In this case the IFBEE are called the water wave equations.

5.1. **Lagrangian coordinates.** As already mentioned, we will rewrite equations (IFBEE) in a fixed domain. This can be done with the help of Lagrangian coordinates, defined as follows.

Let η be the flow of u, i.e., let η solve the ODE

$$\partial_t \eta(t,x) = u(t, \eta(t,x)),$$

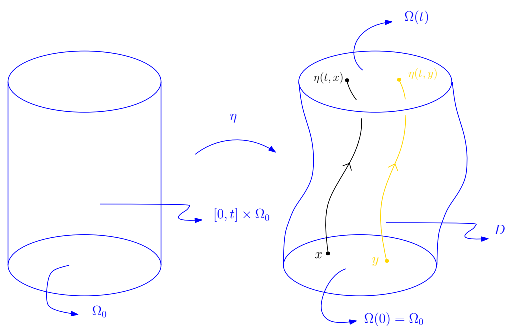
$$\eta(0,x) = x,$$

$x \in \Omega_0$. Then, η is a one-parameter family of volume preserving embeddings of Ω_0 into $\mathbb{R}^4$ (η is the flow of a vector field hence it is a one-parameter family of diffeomorphisms. of Ω_0 onto its image through η; these diffeomorphisms are volume preserving because u is divergence-free). Using η we can write $\Omega(t)$ explicitly as

$$\Omega(t) = \eta(t,\Omega) \equiv \eta(t)(\Omega).$$

This last equation shows exactly how $\Omega(t)$ (or D) depends on the solution since η of course depends on u.

Remark 5.3. Physically, $\eta(t,x)$ corresponds to the position at time t of the fluid particle that at time zero was at x.



We can write the equation defining η as

$$\partial_t \eta = u \circ \eta$$
$$\eta(0) = \mathrm{id},$$

where we henceforth adopt the following notation:

Notation 5.4. When we write composition with η, it always means composition in the spatial variables only. For example, if $f = f(t,x)$, then $f\circ\eta$ means $(f\circ\eta)(t,y) = f(t,\eta(t,y))$.

We can now rewrite equations (IFBEE) in terms of η. They read:

$$\begin{aligned} \partial_t^2\eta + \nabla p\circ\eta &= 0 \quad \text{in} \quad [0,T)\times\Omega_0 \quad \text{(IFBEE-La)},\\ (\mathrm{div}((\partial_t\eta)\circ\eta^{-1}))\circ\eta &= 0 \quad \text{in} \quad [0,T)\times\Omega_0 \quad \text{(IFBEE-Lb)},\\ p\circ\eta &= \sigma\mathscr{H}\circ\eta \quad \text{on} \quad [0,T)\times\partial\Omega_0 \quad \text{(IFBEE-Lc)}, \end{aligned}$$

with initial conditions

$$\begin{aligned} \eta(0,\cdot) &= \mathrm{id} \quad \text{in} \quad \Omega_0 \quad \text{(IFBEE-Ld)},\\ \partial_t\eta(0,\cdot) &= u_0 \quad \text{in} \quad \Omega_0 \quad \text{(IFBEE-Le)}, \end{aligned}$$

where id is the identity diffeomorphism in Ω_0, and η^{-1} is the inverse of the map $x \mapsto \eta(t,x)$, t is fixed. (By construction, η is invertible.)

Equations (IFBEEE-L) are known as the incompressible free-boundary Euler equations in Lagrangian coordinates (abbreviated IFBEE-L). (Equations (IFBEE) are sometimes referred to as the equations in Eulerian coordinates.) η is called the Lagrangian map.

Remark 5.5. Since η is, for each t, a diffeomorphisms between Ω_0 and $\Omega(t)$, it does correspond to a change of coordinates. This change of coordinates however, depends on the solution u. Thus, Lagrangian coordinates are coordinates adapted to the solution. (Compared to Jared's lectures and the discussion of coordinates in GR.)

The advantage of (IFBEE-L) is taht these equations are defined in a fixed domain. The disadvantage is that it introduces complex non-linearities: composition with η, η^{-1}). If η is sufficiently regular, a solution to (IFBEE-L) yields a solution to (IFBEE) upon defining $u = \partial_t \eta \circ \eta^{-1}$.

Remark 5.6. The Lagrangian map and Lagrangian coordinates can be defined for the IEE and the CEE as well.

5.2. **Local existence and uniqueness.** We now restrict ourselves to $n = 3$

Theorem 5.7. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth connected boundary. Let $u_0 \in H^s(\Omega)$ be a divergence-free vector field, where $s > \frac{3}{2} + 2$. Assume that $\sigma > 0$. Then, there exists a T_* and a unique solution (η, p) to (IFBEE-L) defined on the time interval $[0, T]$. The solution satisfies*

$$\eta \in C^0([0,T_*], H^s(\Omega)), \quad \partial_t \eta \in L^\infty([0,T_*], H^s(\Omega))$$
$$\partial_t^2 \eta \in L^\infty([0,T_*], H^{s-\frac{3}{2}}(\Omega)), \quad p \in L^\infty([0,T_*], H^{s-\frac{1}{2}}(\Omega(t)))$$

where $\Omega(t) = \eta(t, \Omega)$, Moreover, $\partial\Omega(t)$ is H^{s+1}-regular.

(Because we work in Lagrangian coordinates, the domain is fixed, thus we wrote Ω for Ω_0 in the statement of the theorem.)

Note the factors $s - \frac{3}{2}$. This comes from the fact, to be explained below, that for the IFBEE ∂_t scales roughly as $\partial x^{\frac{3}{2}}$ (for $\sigma > 0$; all that follows is for $\sigma > 0$).

The statement that $\partial\Omega(t)$ is H^{s+1}-regular says that the boundary is more regular than a naive counting suggests. First, note that since η is in H^s, $\partial\Omega(t)$ will in general not be smooth even if it is smooth at $t = 0$. Indeed, since $\eta \in H^s(\Omega)$, $\eta|_{\partial\Omega} \in H^{s-frac12}(\partial\Omega)$, thus we expect $\partial\Omega(t)$ to be $H^{s-\frac{1}{2}}$ regular. However, using the mean curvature, which gives an elliptic operator, we can improve the boundary regularity. This extra regularity of $\partial\Omega(t)$ is crucial for the proof. (Note, however, that $\mathscr{H}$ will give an equation with Sobolev regular coefficients, so the situation is more complicated than standard elliptic theory.)

Our strategy is to derive a good equation for the motion of the boundary. This is done as follows. Suppose that we can write η as (we will justify this later)

$$\eta = (\mathrm{id} + \nabla f) \circ \beta,$$

where β is a volume preserving diffeomorphism of Ω, and f is a real valued function defined on Ω. In particular, $\beta(\partial\Omega) = \partial\Omega$, so the motion of the boundary is governed by ∇f. Since $J(\beta) = J(\eta) = 1$, where J is the Jacobian,

$$J(\mathrm{id} + \nabla f) \circ \beta) = J(\mathrm{id} + \nabla f) J(\beta) = J(\mathrm{id} + \nabla f) = 1$$

Expanding $J(\mathrm{id} + \nabla f) = 1$ we find

$$\triangle f + \mathscr{N}(f) = 0 \quad \text{in} \quad \Omega$$

where $\mathscr{N}(f)$ contains terms that are quadratic and cubic in $D^2 f$ (the 1 cancels with $J(\mathrm{id})$). If f is small, and we prescribe $f|_{\partial\Omega} = h$, then the problem

$$\begin{aligned} \triangle f + \mathscr{N}(f) &= 0 \quad \text{in} \quad \Omega, \\ f &= h \quad \text{on} \quad \partial\Omega, \end{aligned}$$

(Elliptic-f)

is a perturbation of the Dirichlet problem. Thus, by the implicit function theorem, f is completely determined by its boundary values h. Assuming that $\beta(0) = \mathrm{id}$, we have $\nabla f(0) = 0$. Then by continuity in time, f will be small for small time (of course, this says that $\nabla f(t)$ is small; to go from ∇f to f we work modulo constants. This is an issue that has to be dealt with but we will ignore it in these lectures).

We conclude that it suffices to know $f|_{\partial\Omega}$. Thus, we seek an equation for $f|_{\partial\Omega}$. To do so, roughly, we differentiate $\eta = (\mathrm{id} + \nabla f) \circ \beta$ twice in time, plug it into (IFBEE-La), restrict the resulting expression to $\partial\Omega$, and invoke the boundary (IFBEE-La). We find:

$$\partial_t^2 f - \sigma \mathscr{L}(\partial^2 f, \partial^3) f = F \quad \text{in} \quad [0,T] \times \partial\Omega. \quad \text{(Evol-f)}$$

$\mathscr{L}(\partial^2 f, \partial^3)$ is a third order pseudo-differential operator with coefficients depending on at most second derivatives of f. It comes from the mean curvature ($\mathscr{H}$ is second order in $\eta = (\mathrm{id} + \nabla f) \circ \beta$, thus third order in f). To top order and at the linear level, $\mathscr{L}$ is given by

$$\mathscr{L}_{\text{linear, top}} = \bar{\triangle} \partial \nu$$

where $\bar{\triangle}$ is the Laplacian on $\partial\Omega$ (with respect to the Euclidean metric induced on $\partial\Omega$) and ∂_ν is the normal derivative. ∂_ν depends on the interior values of f, revealing the pseudo-differential nature of $\mathscr{L}$. More precisely, $\bar{\triangle}\partial_\nu$ (and similarly $\mathscr{L}$) is thought of as a Dirlichet-to-Nuemann type of map, as follows. Given a (small) $h : \partial\Omega \to \mathbb{R}$, $\bar{\triangle}\partial_\nu h$ is computed by (i) solving (Elliptic-f), (ii) calculating $\partial_\nu f|_{\partial\Omega}$ (iii) taking the boundary Laplacian of $\partial_\nu f|_{\partial\Omega}$. In particular, equation (Evol-f) has to be solved coupled (Elliptic-f). In particular, equation (Evol-f) has to be solved coupled to (Elliptic-f).

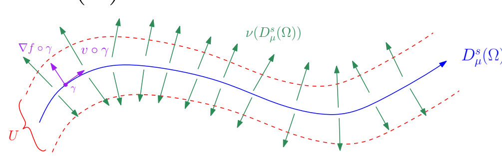
The term F in (Evol-f) contains lower order terms (whose form matters but will not be discussed here).

We think of (Evol-f) as a wave-like equation on $\partial\Omega$, where the Laplacian has been replaced by $\mathscr{L}$. Equation (Evol-f) is the crucial equation to establish the theorem.

Remark 5.8. Below, we will repeatedly use the fact that a vector field in Ω can be decomposed as a divergence-free and tangent to $\partial\Omega$ part and a gradient part.

Sketch of the proof of the theorem: Let $P(u_0)$ be the projection of u_0 onto its divergence-free and tangent to the boundary part (thus, we subtract from u_0 its normal component). Let ζ be the Lagrangian flow of the solution to the IEE in Ω with initial data $P(u + 0)$. Set $\eta_0 = \beta_0 = \zeta$, $f_0 = 0$, and define $\{\eta_\ell\}$, $\{\beta_\ell\}$, $\{f_\ell\}$ inductively as follows.

Step 1. Let ν_ℓ be a given curve of H^s embedding of Ω into $\mathbb{R}^3$ such that $\eta(0) = \mathrm{id}$. Let $D^s_\mu(\Omega)$ be the space of volume-preserving diffeomorphisms of Ω. It is a fact that $D^s_\mu(\Omega)$ is an infinite dimensional Riemannian manifold that has a smooth normal bundle $\nu(D^s_\mu(\Omega))$ inside $H^s(\Omega)$ and a smooth exponential map that maps $\nu(D^s_\mu(\Omega))$ diffeomorphically onto a neighborhood U of $D^s_\mu(\Omega)$ in $H^s(\Omega)$.



A tangent vector at $\gamma \in D^s_\mu(\Omega)$ is given by $v \circ \gamma$, with $\mathrm{div}(v) = 0$, and a normal vector by $\nabla g \circ \gamma$, for some vector field v and some function g. Therefore, we conclude that if η_ℓ is sufficiently close to $D^s_\mu(\Omega)$ (thus if time is small), there exists a function g_ℓ and a $\gamma_\ell \in D^s_\mu(\Omega)$ such that $\eta_\ell = (\mathrm{id} + \nabla g_\ell) \circ \gamma_\ell$. Set $\beta_{\ell+1} = \gamma_\ell$.

Remark 5.9. Note that we do not take f from this decomposition. This is because g_ℓ has no connection with the boundary condition.

Technical note. $T_\gamma(D^s_\mu(\Omega))$ is given by elements of the form $v \circ \gamma$, with $\mathrm{div}(v) = 0$ If $v, w \in T_\gamma(D^s_\mu(\Omega))$ their inner product is given by the L^2 inner product $\langle v, w\rangle = \int\limits_\Omega (v \circ \eta) \cdot (w \circ \eta)$. The normal bundle is normal in the L^2 sense , and at γ and it is given by $\nabla_\gamma \triangle_\gamma^{-1} \mathrm{div}_\gamma(H^s(\Omega))$, which is smooth in γ since $\mathrm{div}_\mu : H^s \mapsto H^{s-1}$ and $\nabla_\gamma \triangle_\gamma^{-1} : H^{s-1} \mapsto H^s$ are both smooth in γ (see [18]). Thus, the normal bundle is smooth in H^s even though it is normal only in the L^2 sense (similarly for the exponential map). Above L_γ is defined as $L_\gamma(z) = (L(z \circ \gamma^{-1})) \circ \gamma$.

Step 2.

$$
\begin{aligned}
\triangle p_{\mathrm{int},\,\ell+1} &= -\mathrm{div}(\nabla_{\hat{u}_\ell} \hat{u}_\ell) \quad \text{in} \quad (\mathrm{id} + \nabla f_\ell) \circ \beta_\ell(\Omega), \\
p_{\mathrm{int},\,\ell+1} &= 0 \quad \text{on} \quad \partial(\mathrm{id} + \nabla f_\ell) \circ \beta_\ell(\Omega),
\end{aligned}
$$

where $\hat{u}_\ell$ is a divergence-free vector field in $(\mathrm{id} + \nabla f_\ell) \circ \beta_\ell(\Omega)$ constructed out of the inductive quantities at step ℓ (see below). The idea for the equation for $p_{\mathrm{int},\ell+1}$ is that we can write $p = p_{\mathrm{int}} + \sigma p_*$, where p_{int} is zero on the boundary and p_* is the harmonic extension of the mean curvature. (Compare to the IEE.)

Let us comment on the reason to introduce $\hat{u}_\ell$ (whose definition is given below). η_ℓ is an embedding that is not necessarily volume preserving (see the definition of $\eta_{\ell+1}$ below). Thus, while $\partial_t \eta_\ell = \tilde{u}_\ell \circ \eta_\ell$ for some vector field $\tilde{u}$ defined in $\eta_\ell(\Omega)$, $\mathrm{div}(\tilde{u}_\ell) = 0$ may not hold (note that we did not say in step 1 that η_ℓ is volume preserving). We do need a divergence-free vector field though in order to get the correct regularity for $p_{\mathrm{int},\ell+1}$ (we need $\mathrm{div}(\nabla_{\hat{u}} \hat{u})$ not to involve second derivatives of $\hat{u}$). We have, therefore to "correct" $\partial_t \eta_\ell$ by constructing an appropriate diverge-free vector field in the domain $(\mathrm{id} + \nabla f_\ell) \circ \beta_\ell(\Omega)$. Note also, that the domains $(\mathrm{id} + \nabla f_\ell) \circ \beta_\ell(\Omega)$ and $\eta_\ell(\Omega)$ might not be equal (again, see the definition of $\eta_{\ell+1}$ below).

Step 3. Using $\beta_{\ell+1}$ and $p_{\mathrm{int},\ell+1}$ into (Evol-f), we solve the corresponding linear equation for f with initial conditions $f(0, \cdot) = 0$, and $\partial_t f(0, \cdot) = z|_{\partial\Omega}$, where z solves

$$
\begin{aligned}
\triangle z &= \mathrm{div}(u_0) \quad \text{in} \quad \Omega \\
\frac{\partial z}{\partial \nu} &= u_0 \cdot \nu \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

We call the solution $f_{\ell+1}$. (We comment below on how to solve equation (Evol-f) which, we recall, is solved coupled to (Elliptic-f)).

Step 4. Define $h_{\ell+1}$ by solving

$$
\begin{aligned}
&\triangle h_{\ell+1} = 0 \quad \text{in} \quad (\mathrm{id} + \nabla f_{\ell+1})(\Omega) \\
&\frac{\partial h_{\ell+1}}{\partial \tilde{\nu}_{\ell+1}} = \big((\nabla \partial_t f_{\ell+1} + \nabla_{v_{\ell+1}} f_{\ell+1} + v_{\ell+1}) \circ (\mathrm{id} + \nabla f_{\ell+1})^{-1} \big) \cdot \tilde{\nu}_{\ell+!} \quad \text{on} \quad \partial(\mathrm{id} + \nabla f_{\ell+1})(\Omega),
\end{aligned}
$$

where $v_{\ell+1}$ is defined by $\partial_t \beta_{\ell+1} = v_{\ell+1} \circ \beta_{\ell+1}$ and $\tilde{\nu}_{\ell+1}$ is the outer unit normal to $\partial(\mathrm{id} + \nabla f_{\ell+1})(\Omega)$.

To motivate this equation note that if we have a solution to the IFBEE, we can decompose u in its divergence-free and tangent to the boundary part and its gradient part: $u = p(u) + \nabla h$. Taking divergence we see that h is harmonic. Using $\eta = (\mathrm{id} + \nabla f) \circ \beta$ and $\partial_t \eta = u \circ \eta$, we can compute $\frac{\partial h}{\partial \nu}$ in terms of f and β, which gives the above boundary condition.

Step 5. Define $\bar{\eta}_{\ell+1} = (\mathrm{id} + \nabla f_{\ell+1}) \circ \beta_{\ell+1}$ (this is not yet $\eta_{\ell+1}$). By construction it is volume preserving thus the velocity $\bar{u}^{\ell+1}$ given by $\partial_t \eta_{\ell+1} = u_{\ell+1} \circ \bar{\eta}_{\ell+1}$ is divergence free. Define a vector field $z_{\ell+1}$ in Ω by solving

$$\partial_t z_{\ell+1} = Q_{\bar{\eta}_{\ell+1}}((\nabla_{\bar{u}_{\ell+1}})_{\bar{\eta}_{\ell+1}}(z_{\ell+1}) - P_{\bar{\eta}_{\ell+1}}((\nabla_{z_{\ell+1} \circ \bar{\eta}^{-1}_{\ell+1}})_{\bar{\eta}_{\ell+1}}(\nabla h_{\ell+1} \circ \bar{\eta}_{\ell+1})) + \nabla H_{\ell+1} \circ \bar{\eta}_{\ell+1}$$
$$\text{in} \quad [0,T] \times \Omega$$

with initial condition $z(0) = P(u_0)$. P and Q are, respectively, the operators that project a vector field onto its divergence-free and tangent to the boundary part and its gradient part in the domain $\bar{\eta}_{\ell+1}(\Omega)$. The operators "sub-$\bar{\eta}_{\ell+1}$" are defined as follows. If L is an operator acting on maps defined in $\bar{\eta}_{\ell+1}(\Omega)$, then $L_{\bar{\eta}_{\ell+1}}$ acts on maps ω defined in Ω by $L_{\bar{\eta}_{\ell+1}}(\omega) = \left(L\left(\omega \cdot \bar{\eta}^{-1}_{\ell+1}\right)\right) \circ \bar{\eta}_{\ell+1}$. Finally, $H^{\ell+1}$ solves

$$\triangle H_{\ell+1} = 0 \quad \text{in} \quad \bar{\eta}_{\ell+1}(\Omega),$$
$$\frac{\partial H_{\ell+1}}{\partial \bar{\nu}_{\ell+1}} = \left(z_{\ell+1} \circ \bar{\eta}^{-1}_{\ell+1}\right) \cdot (\partial_t(\bar{\nu}_{\ell+1} \circ \bar{\eta}_{\ell+1})) \circ \bar{\eta}^{-1}_{\ell+1} + (\nabla_{\bar{u}_{\ell+1}}(z_{\ell+1} \circ \bar{\eta}_{\ell+1})) \cdot \bar{\nu}_{\ell+1} \quad \text{on} \quad \partial\bar{\eta}_{\ell+1}(\Omega).$$

The idea for finding the equation for $z_{\ell+1}$ is to see which equation $P(U)$ satisfies, and then reinterpret it from the point of view of the iteration.

Technical note. $\nabla H_{\ell+1}$ appears because ∂_t and the projection P (both with respect to the moving domain) do not commute.

It can be showed that the equation for z can be written as an ODE in an $P_{\bar{\eta}_{\ell+1}}\left(PH^{s-1}(\bar{\eta}_{\ell+1}(\Omega))\right)$. After finding a solution $z^{\ell+1}$, we can estimate $\mathrm{curl}(\bar{z}_{\ell+1} \circ \bar{\eta}^{-1}_{\ell+1})$ to show that $z_{\ell+1}$ is in fact H^s-regular.

The above steps define $f_{\ell+1}$, $\beta_{\ell+1}$, $h_{\ell+1}$, and $z_{\ell+1}$. We now set

$$\eta_{\ell+1} = \mathrm{id} + \int_0^t (z_{\ell+1} + \nabla h_{\ell+1} \circ (\mathrm{id} + \nabla f_{\ell+1}) \circ \beta_{\ell+1}).$$

With this definition we have

$$\partial_t \eta_{\ell+1} = \left(z_{\ell+1} \circ ((\mathrm{id} + \nabla f_{\ell+1}) \circ \beta_{\ell+1})^{-1} + \nabla h_{\ell+1}\right) \circ (\mathrm{id} + \nabla f_{\ell+1}) \circ \beta_{\ell+1},$$

from which we define

$$\hat{u}_{\ell+1} = z_{\ell+1} \circ ((\mathrm{id} + \nabla f_{\ell+1}) \circ \beta_{\ell+1})^{-1} + \nabla h_{\ell+1}.$$

With the above sequences at hand, the next steps are as follows:

a) We use the several equations introduce above (including (Evol-f)) to obtain estimates that can be used to show that the sequences $\{\eta_\ell\}$, $\{\beta_\ell\}$, etc. converge.

b) We show that the limit quantities satisfy equations (IFBEE-L). To do so, we use that the pressure has an "interior" and a "boundary" part (see Step 2). The boundary part is given by the harmonic extension of the mean curvature, hence it is determined by the geometry. The latter, in turn, is constructed out of the sequence of domains $(\mathrm{id} + \nabla f_\ell) \circ \beta_\ell$.

Let us make a brief comment on how equation (Evol-f) is treated. The (very) rough idea is that we think of (Evol-f) as a wave equation thus we multiply it by $\partial_t f$ and integrate by parts. But since $\mathscr{L}$ is third order, we integrate by parts $\frac{3}{2}$ derivatives, obtaining an estimate of the type

$$\|\partial_t f\|_{0,\partial}^2 + \sigma\|\partial^{\frac{3}{2}} f\|_{0,\partial}^2 \lesssim (\text{initial data} + \int_0^t \|F\|_{0,\partial}.$$

As in the case of wave equations, we can also derive higher order energy estimates. Such estimates are then used to construct solutions to (Evol-f).

Finally, let use comment on the regularity of $\partial\Omega(t)$ stated in the theorem. Since $\beta(\partial\Omega) = \partial\Omega$, the regularity of the boundary is determined by the regularity of ∇f. From the above estimate for f, we see that f gains $\frac{3}{2}$ derivatives with respect to the source F. Using this gain of regularity, we can show that ∇f is $\frac{3}{2}$ derivatives more regular than u/ Thus, $\nabla f/\partial\Omega$ is in $H^{s+1}(\partial\Omega)$ if $u \in H^s(\Omega)$, which gives that $\partial\Omega(t)$ is H^{s+1}-regular.

Technical note. Even though F involves ∇u, it also involves $\triangle_\nu^{-1}$, so the regularity of F is not what we get from a naive derivative counting.

This extra regularity of the boundary is very important for the proof. As seen, we are solving several elliptic boundary value problems in our constructions, and for this we need the boundary to be sufficiently regular.

We refer to [12] for details of the proof. $\square$

From the above discussion, we see that, upon closing the estimates for f, we will get an estimate of the type

$$\|\partial_t f\|_{s,\partial} + \sigma\|f\|_{s+\frac{3}{2},\partial} \leq C.$$

This suggests that if σ is very large then f is very small. Since f controls the motion of the boundary, it means $\partial\Omega(t)$ has small amplitude. This is the content of the following theorem.

Theorem 5.10. *(informal version) When $\sigma \to \infty$, solutions to the IFBEE converge to solutions of the IEE in the fixed domain Ω.*

See [12] *for a precise statement, as well as for a discussion of how $\sigma \to \infty$ corresponds to a well-studied situation of constrained motion in mechanics (see also* [11]*).*

The above proof involves several ideas that differ considerably then the standard approach illustrated by quasi-linear wave equations. There are other ways to approach the IFBEE (see references in [12]) but the point to keep in mind is that essential new ingredients (as compared to known approaches to study fluid equations) are needed to treat the IFBEE.

We had mentioned that not only the construction of solutions for the free-boundary Euler equations is significantly different that the fluid equations in a fixed domain (or than quasi-linear wave equations), but that the method for deriving a priori estimates also contains key new aspects. We will illustrate this next for the compressible free boundary Euler equations.

Remark 5.11. When $\sigma = 0$, the IFBEE are ill-posed. However, they are locally well posed if $p(0)$ satisfies $\frac{\partial p(0)}{\partial \nu} \leq$ constant < 0 on $\partial\Omega$, known as Taylor-sign condition. This condition can be thought of as a physical condition (p should be positive in the interior).

6. The compressible free-boundary Euler equations

The compressible free-boundary Euler equations (CFBEE) are given by

$$\partial_t u + \nabla_u u + \frac{1}{\rho}\nabla p = 0 \quad \in \quad D \quad \text{(CFBEEa)},$$
$$\partial_t \rho + \nabla_u \rho + \rho \operatorname{div}(u) = 0 \quad \text{in} \quad D \quad \text{(CFBEEb)},$$

$$p = p(\rho) \quad \text{in} \quad D \quad \text{(CFBEEc)},$$
$$p = \sigma \mathscr{H} \quad \text{on} \quad \partial D \quad \text{(CFBEEd)},$$
$$\partial_t + u^i \frac{\partial}{\partial x^i} \in T(\partial D) \quad \text{(CFBEEe)},$$

where $D = \bigcup_{0 \leq t < T} \{t\} \times \Omega(t) \quad$ (CFBEEf), with initial conditions

$$u(0, \cdot) = u_0 \quad \text{(CFBEEg)},$$
$$\rho(0, \cdot) = \rho_0 \quad \text{(CFBEEh)},$$
$$\Omega(0, \cdot) = \Omega_0 \quad \text{(CFBEEi)}.$$

The meaning of all quantities in (CFBEE) is as in equations (CEE) and (IFBEE). We will henceforth assume to be working in three spatial dimensions, i.e., $n = 3$.

As we did for the IFBEE, we will rewrite the equations in the fixed domain Ω_0 by introducing Lagrangian coordinates.

Let η be the flow of u and define

$$v = \partial_t \eta, \quad R = \rho \circ \eta, \quad q = p \circ \eta.$$

(we are using the same notation for composition as done for the IFBEE, i.e., composition is on the spatial variables only.) v, R, and q are called, respectively, the Lagrangian velocity, Lagrangian density, and Lagrangian pressure. We will write Ω for Ω_0 from now on. In terms of v, R, and q, equations (CFBEE) read:

$$R \partial_t v^\alpha + a^{\mu\alpha} \partial_\mu q = 0 \quad \text{in} \quad [0, T) \times \Omega \quad \text{(CFBEE-La)},$$
$$\partial_t R + R a^{\mu\alpha} \partial_\mu v_\alpha = 0 \quad \text{in} \quad [0, T) \times \Omega \quad \text{(CFBEE-Lb)},$$
$$\partial_t a^{\alpha\beta} + a^{\alpha\gamma} \partial_\mu v_\gamma a^{\mu\beta} = 0 \quad \text{in} \quad [0, T) \times \Omega \quad \text{(CFBEE-Lc)},$$
$$\eta = \mathrm{id} + \int_0^t v \quad \text{in} \quad [0, T) \times \Omega \quad \text{(CFBEE-Ld)},$$
$$q = q(R) \quad \text{in} \quad [0, T) \times \Omega \quad \text{(CFBEE-Le)},$$
$$q a^{\mu\alpha} N_\mu + \sigma (a^T N) \triangle_g \eta^\alpha = 0 \quad \text{on} \quad [0, T) \times \partial\Omega \quad \text{(CFBEE-Lf)},$$
$$v(0, \cdot) = u_0 \quad \text{in} \quad \Omega \quad \text{(CFBEE-Lg)},$$
$$R(0, \cdot) = \rho_0 \quad \text{in} \quad \Omega \quad \text{(CFBEE-Lh)}.$$

The notation/variables are as follows. Greek indices run from 1 to 3. Indices will be raised and lowered with the Euclidean metric. $a^{\alpha\beta}$ is a matrix given by $a = (D\eta)^{-1}$, where $(D\eta)^{-1}$ is the inverse of the matrix $D\eta$ (recall that D denotes derivatives in the spatial variables only). N is the outer normal to $\partial\Omega$. a^T is the transpose of the matrix a. $\triangle_g$ is the Laplacian on $\partial\Omega$ with respect to the metric g induced on the boundary by the embedding η. In coordinates such that $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$ are tangent to $\partial\Omega$, g reads

$$g^{ij} = \partial_i \eta^\mu \partial_j \eta_\mu, \quad i, j = 1, 2,$$

and $\triangle_g$ is given by

$$\triangle_g(\cdot) = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j(\cdot))$$

(Latin indices vary from 1 to 2.) $|g|$ is the determinant of g, and g^{ij} is the inverse of g. In (CFBEE-L f), $\triangle_g$ acts componentwise on η. Observe that from (CFBEE-Ld) we have $\eta(0) = \mathrm{id}$, so that $a(0) = I$=identity matrix.

Remark 6.1. Compared to (IFBEE-L), here we have chosen to write the equations in a way that avoids carrying right composition with η (such as $\nabla p \circ \eta$ in (IFBEE-La)), which can be done by introducing the Lagrangian density and pressure. This leads to the presence of the matrix a.

Remark 6.2. As in the case of equations (CEE), the initial data for the CFBEE has to satisfy compatibility conditions.

The following are two important identities for solutions of (CFBEE-L):

The first identity is as follows. Let

$$J = \det(D\eta).$$

Note that $J > 0$ for small time. Then

$$RJ = \rho_0. \quad \text{(density-J)}$$

For the second identity, define

$$A = Ja.$$

Then

$$\partial_\mu A^{\mu\alpha} = 0, \quad \alpha = 1, 2, 3. \text{ (Piola)}$$

This last identity is known as Piola's identity (see [20], chapter 8).

The CFBEE behave differently depending on the following distinctions.

value of σ/bound on ρ_0	$= 0$ (no surface tension)	> 0 (with surface tension)
$\rho_0 \leq \lambda > 0$ (liquid)	(a) liquid with no surface tension	(b) liquid with surface tension
ρ_0 allowed to vanish (gas)	(c) gas with no surface tension	(d) gas with surface tension

Cases (a), (b), (c), and (d) are different not only with respect to their physical content but also regarding the techniques used to study them. Here we will focus on case (b) (all cases, perhaps with exception of (d), are physically relevant).

It is sometimes convenient to write (CFBEE-La) more explicitly in terms of R (upon using (CFBEE-Le))

$$R\partial_t v^\alpha + q'(R) a^{\mu\alpha} \partial_\mu R = 0 \quad \text{in} \quad [0, T) \times \Omega \quad \text{(CFBEE-La')}$$

It is also convenient to multiply (CFBEE-Lf) by J obtaining

$$qA^{\mu\alpha} N_\mu + \sigma \sqrt{|g|} \triangle_g \eta^\alpha = 0 \quad \text{in} \quad [0, T) \times \partial\Omega, \quad \text{(CFBEE-Lf')}$$

where we used the identity $J|a^T N| = \sqrt{|g|}$.

6.1. **An attempt at a priori estimates.** We now turn to the question of a priori estimates for equations (CFBEE-L). Let us starting asking what kind of regularity we can naively expect. Heuristically, equations (CFBEE-La') and (CFBEE-Lb) suggest $\partial_t v \sim \nabla R$ and $\partial_t R \sim \nabla v$. Based also on our experience with the CEE, we naively expect $\partial_t \sim \nabla$ (i.e. one time derivatives correspond to one spacial derivative, differently than what happens to the IFBEE). And taking again (CEE) as motivation, we expect to be able to close estimates at the same regularity level for v and R. Therefore, we seek to close the estimates with:

$$v \in H^s, \quad R \in H^s, \quad \partial_t v \in H^{s-1}, \quad \partial_t R \in H^{s-1}.$$

(The ideas in the last paragraph are very heuristic and should be taken only as a vague motivation. Our "guess" for the regularity properties of solutions relies more on hindsight and experience than anything else.)

Let us now see what difficulties arise if we try the standard approach for deriving a priori estimates (recall the previous discussions). Let us write ~ 0 to mean modulo terms that are not top order in derivatives. Taking D^s of (CFBEE-La') and contracting with $D^s v_\alpha$:

$$RD^s v_\alpha \partial_t D^s v^\alpha + q'(R) D^s v_\alpha a^{\mu\alpha} \partial_\mu D^s R \sim 0$$

We will integrate over Ω, but we see that $\partial_\mu D^s R$ contains too many derivatives. Thus we need to integrate ∂_μ by parts. To avoid picking extra derivatives when doing so it is convenient to multiply by J so that we can use (Piola). Thus

$$\int_\Omega JRD^s v_\alpha \partial_t D^s v_\alpha + \int_\Omega q'(R) D^s v_\alpha A^{\mu\alpha} \partial_\mu D^s R \sim 0$$

Integrating ∂_μ by parts in the second integral and using (Piola):

$$\int_\Omega D^s v_\alpha A^{\mu\alpha} \partial_\mu D^s R = -\int_\Omega q'(R) A^{\mu\alpha} \partial_\mu D^s v_\alpha D^s R + \int_{\partial\Omega} q'(R) D^s v_\alpha A^{\mu\alpha} N_\mu D^s R. \quad \text{(CFB-trial 1)}$$

From (CFBEE-Lb)

$$\partial_t D^s R + R a^{\mu\alpha} \partial_\mu D^s v_\alpha \sim 0$$

so that

$$\int_\Omega D^s v_\alpha A^{\mu\alpha} \partial_\mu D^s R \sim \int_\Omega \frac{q'(R)}{R} J \overbrace{\partial_t D^s R D^s R}^{\frac{1}{2}\partial_t (D^s R)^2} + \int_{\partial\Omega} q'(R) D^s v_\alpha A^{\mu\alpha} N_\mu D^s R.$$

Using this last expression and (density-J) into (CFB-trial 1)

$$\frac{1}{2}\partial_t \int_\Omega \rho_0 |D^s v|^2 + \frac{1}{2}\partial_t \int_\Omega \frac{q'(R)}{R} J (D^s R)^2 + \int_{\partial\Omega} q'(R) D^s v_\alpha A^{\mu\alpha} N_\mu D^s R \sim 0. \quad \text{(CFB-trial 2)}$$

The first two terms will produce (to top order) $\|v\|_s^2$ and $\|R\|_s^2$ after integration in time (note that $\frac{q'(R)J}{R} > 0$ under reasonable assumptions on the equation of state). Therefore, to close the estimates, we need to bound the boundary term in terms of $\|v\|_s$ and $\|R\|_s$. Let us look at this term more closely.

An immediate difficulty is the following: since we want to bound the boundary integral by $\|v\|_s$ and $\|R\|_s$, we need to bound $D^s v|_{\partial\Omega}$ and $D^s R|_{\partial\Omega}$ by $\|v\|_s$ and $\|R\|_s$. This does not seem to be directly possible: even using the most "economic" inequality (in the sense that it does not add any derivatives to v or R),

$$\int_{\partial\Omega} q'(R) D^s v_\alpha A^{\mu\alpha} N_\mu D^s R \leq \|q'(R) A\|_{L^\infty} \|D^s v\|_{0,\partial} \|D^s R\|_{0,\partial}$$

we cannot find the desired bound since there is no general inequality of the form

$$\|f\|_s \leq C \|D^s f\|_{0,\partial}$$

(said differently, for generic $f \in H^s$, $D^s f \in L^2$ but no better, so $D^s f|_{\partial\Omega}$ might not be well-defined).

Our only hope seems here is to use that v and R satisfy (CFBEE-L) and use the structure of these equations. It is natural to invoke the boundary condition (CFBEE-L f'). For this, we revert back to q:

$$q'(R)D^sR \sim D^sq$$

so that the boundary integral becomes

$$\int_{\partial\Omega} D^s v_\alpha A^{\mu\alpha} N_\mu D^s q.$$

We now invoke (CFBEE-L f') to get

$$A^{\mu\alpha}N_\mu D^s q \sim -\sigma\sqrt{|g|}\triangle_g D^s\eta^\alpha, \quad \text{(CFB-trial 3)}$$

so that the boundary integral is now:

$$-\sigma\int_{\partial\Omega}\sqrt{|g|}D^s v_\alpha \triangle_g D^s \eta^\alpha. \quad \text{(see comments about tangential derivatives further below)}$$

This is still not good enough since we still cannot bound $D^s v$ and we seem to have too many derivatives. At this point we might suspect that we cannot close the estimates by taking D^s of the equations and proceeding as in usual cases.

Remark 6.3. Another problem with the above argument is the following. When we wrote the equations ~ 0, we indicated the most obvious top order terms. But there are other terms that also contribute to top order that have been omitted and need to be handled.

6.2. **A different approach.** Suppose that in the above argument, instead of D^s we use $\partial_t D^{s-1}$ (this is consistent with our expected regularity in view of $\partial_t \sim \nabla$). Then the boundary term becomes

$$-\sigma\int_{\partial\Omega}\sqrt{|g|}\partial_t D^{s-1}v_\alpha \triangle_g \partial_t D^{s-1}\eta^\alpha.$$

We now use that $\partial_t \eta = v$ (so that $\partial_t D^{s-1}\eta = D^{s-1}v$), and integrate by parts the Laplacian to get

$$-\sigma\int_{\partial\Omega}\sqrt{g}\partial_t D^{s-1}v_\alpha\triangle_g\partial_t D^{s-1}\eta^\alpha \sim \sigma\int_{\partial\Omega}\sqrt{g}\partial_t\nabla_g D^{s-1}v\cdot\nabla_g D^{s-1}v \sim \frac{1}{2}\sigma\partial_t\int_{\partial\Omega}|\nabla_g D^{s-1}v|^2.$$

(∇g is the covariant derivative in the metric g.)

Returning to (CFB-trial 2) with the change $D^s \mapsto \partial_t D^{s-1}$, we have

$$\frac{1}{2}\partial_t\int_\Omega \rho_0|\partial_t D^{s-1}v|^s + \frac{1}{2}\partial_t\int_\Omega\frac{q'(R)}{R}J(\partial_t D^{s-1}R)^2 + \frac{1}{2}\sigma\partial_t\int_{\partial\Omega}|\nabla_g D^{s-1}v|^2 \sim 0. \quad \text{(energy-trial)}$$

This basically says that we can control $\|\partial_t v\|^2_{s-1}$, $\|\partial_t R\|^2_{s-1}$, and $\|D^{s-1}v\|^2_{1,\partial}$ (notice the importance of having the correct sign in (CFB-trial 3)). However, a closer look reveals that there is a problem with this argument. In (CFB-trial 3), we differentiated the boundary condition. Naturally, we can only do this if the derivatives are tangential to the boundary. Therefore, instead of $\partial_t D^{s-1}$ we need to use $\partial_t \bar{D}^{s-1}$ where we use the following notation:

Notation 6.4. We will use $\bar{D}$ to denote derivatives constructed with vector field that are tangent to $\partial\Omega$. For example, if Ω is given by $\{x^3 >\}$, then we can take $\bar{D} = \partial_1$ or ∂_2. If Ω is a ball, we can take $\bar{D} = \partial_\theta$ or ∂_ϕ (properly smoothed out near the origin).

There is another important point that requires attention. When we differentiate the boundary condition (now using $\bar{D}$), we produce extra terms, omitted above, that are not lower order. For example, when we commute

$$\partial_t \bar{D}^{s-1}(\triangle_g \eta^\alpha) \sim \triangle_g \partial_t \bar{D}^{s-1}\eta^\alpha$$

we have terms where all derivatives fall on the coefficients of $\triangle_g$ since the coefficients of $\triangle_g$ involve one derivative of g and g involves one derivative of η, we obtain terms of the form

$$\partial_t \bar{D}^{s-1}(\bar{D}^2\eta) = \bar{D}^{s+1}v$$

which have too many derivatives on v (recall we want to bound v in H^s). In order to get around this difficulty, we need to rewrite the boundary condition in a different way and invoke several geometric aspects of the problem. The net effect will be that we will not be able to bound $\|\bar{D}^{s-1}v\|_{1,\partial}$ as suggested above but only the corresponding normal component, i.e., we will get an estimate for $\|\bar{D}^{s-1}v\cdot N\|_{\frac{1}{2},\partial}$ or, alternatively, $\|v\cdot N\|_{s-\frac{1}{2},\partial}$.

We make two more observations.

First, the interior bounds we discussed above become, under the change $D\mapsto \bar{D}$, estimates for $\|\partial_t\bar{D}^{s-1}v\|_0^2$ and $\|\partial_t\bar{D}^{s-1}R\|_0^2$. Since $\bar{D}^{s-1}$ does not involve all derivatives of order s but only those that are tangential to $\partial\Omega$, $\|\partial_t\bar{D}^{s-1}v\|_0^2$ and $\|\partial_t\bar{D}^{s-1}T\|_0^2$ do not give control over $\|\partial_t v\|_{s-1}^2$ and $\|\partial_t R\|_{s-1}^2$ (as suggested when we had D^{s-1}).

Second, we can repeat the above reasoning $\partial_t^2\bar{D}^{s-2}$, ∂_t^3, etc., i.e., all operators of the form $\partial_t^k\bar{D}^{s-k}$, $1\leq k\leq s$.

Summing up, we have concluded the following.

- We need to differentiate the equations with operators of the form $\partial_t^k\bar{D}^{s-k}$, where $1\leq k\leq s$. We need k to be at least one because otherwise we cannot control the boundary integral. In particular, we need to time differentiate the equations. This is a different situation than what we have in, say, quasi-linear wave equations or fluids in a fixed domain.
- This procedure produces estimates (with $1\leq k\leq s$) for:

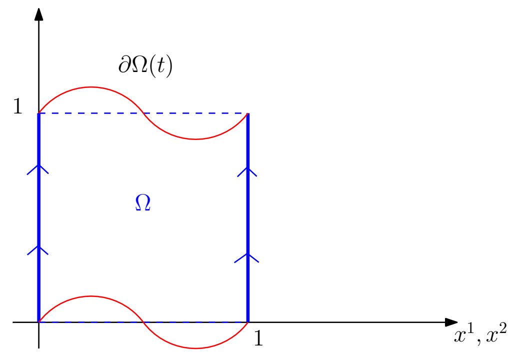
  $$\|\partial_t^k\bar{D}^{s-k}v\|_0,\quad \|\partial_t^k\bar{D}^{s-k}R\|_0,\quad \text{and}\quad \|\partial_t^{k-1}v\cdot N\|_{s-k+1,\partial}.$$

  These estimates are significantly weaker than what we want (recall that we want to bound $\|v\|_s$ and $\|R\|_s$). We will see next how these estimates can be improved to give the bounds that we want.

Remark 6.5. The estimate for $\partial_t^{k-1}v\cdot N$ is in fact slightly different then what was stated above when $k=1$, see below. For $k\leq 2$, however, we note that $\|\partial_t^{k-1}v\cdot N\|_{s-k+1,\partial}$ is a better estimate than what one obtains by restricting $\partial_t^{k-1}v$ to $\partial\Omega$, which gives $\|\partial_t^{k-1}v\|_{s-k+\frac{1}{2},\partial}$ for $\partial_t^{k-1}v\in H^{s-k+1}(\Omega)$. Thus, the normal component of v has better regularity than what we initially expect (a feature due to the mean curvature).

6.3. **A priori estimates.** Here we will derive a priori estimates for equations (CFBEE-L). We will make the following simplifying assumptions (they can be removed with some extra work).

Let us assume that $\Omega = \Pi^2\times(0,1)$, with coordinates (x^1,x^2,x^3). Thus, the motion of the boundary is in the vertical direction and we can take as tangential differential operators $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$.



Sometimes we impose the boundary condition $\sigma \cdot N = 0$ on the bottom boundary $\{x^3 = 0\}$, so that the problem resembles the problem of waves in the ocean.

We also make the following assumptions on the equation of state. Besides the assumptions we had for the CEE, we assume that for some $[a, b], a > 0$, such that $\rho_0(x) \in [a, b]$ for all $x \in \Omega$, we have, for all $r \in [a, b]$, $q'(r) > A$ and $\left(\frac{q(r)}{r}\right)' > A$, for some constant $A > 0$.

Theorem 6.6. *Let v_0 be a smooth vector field in Ω and ρ_0 be a smooth positive function on Ω (bounded away from zero). Let Ω and $q(R)$ be as above and assume $\sigma > 0$. Then there exists $T_* > 0$ and a constant $C_* > 0$, depending only on*

$$\|v_0\|_3, \|v_0\|_{3,\partial}, \|\rho_0\|_3, \|\rho_0\|_{3,\partial}, \sigma, \text{ and } \|(\triangle \mathrm{div}(v_0)\|_{-1,\partial},$$

such that any smooth solution (v, R) to (CFBEE-L) with initial data (v_0, ρ_0) and defined on $[0, T_*]$ satisfies

$$\|v\|_3 + \|\partial_t v\|_2 + \|\partial_t^2 v\|_1 + \|\partial_t^3 v\|_0 + \|R\|_3 + \|\partial_t R\|_2 + \|\partial^2 R\|_2 + \|\partial_t^3 R\|_0 \leq C_*$$

for $0 \leq t \leq T_*$.

Remark 6.7. Since we hope to control $\partial_t^k v \cdot N$with more regularity than what is given by $\partial_t^k v|_{\partial\Omega}$ (see above), we naturally need the initial data to be compatible with such regularity. This assumption on $\triangle \mathrm{div}(v_0)|_{\partial\Omega}$ encodes, in a simple fashion, such extra regularity. (There are further technical conditions for such assumptions as well.)

Before giving a proof, let us state the following proposition that will be needed.

Proposition 6.8. *(Compressible Cauchy invariance) Let (v, R) be a smooth solution (CFBEE-L) defined on the time interval $[0, T]$. Then the following identity holds*

$$\epsilon^{\alpha\beta\gamma}\partial_\beta v_\mu \partial_\gamma \eta^\mu = \omega_0^\alpha + \int_0^t \frac{1}{R}\epsilon^{\alpha\beta\gamma} a^{\lambda\mu}\partial_\lambda q \partial_\beta R \partial_\gamma \eta_\mu, \qquad \textit{(Cauchy-inv)}$$

where $\epsilon^{\alpha\beta\gamma}$ is the totally anti-symmetric tensor (with $\epsilon^{123} = 1$) and ω_∂ is $\mathrm{curl}(v(0))$ (i.e., the vorticity at time zero).

Since $\eta(0) = \mathrm{id}$, $\partial_\gamma \eta^\mu \approx \delta_\gamma^\mu$, thus the LHS of (Cauchy-inv) is roughly $\mathrm{curl}(v)$. This identity therefore says that we can control $\mathrm{curl}(v)$ by its initial value plus a time integral of the fluid variables. From the point of view of closing estimates, the time integral is harmless because we can apply Gronwall's inequality.

For incompressible fluids, identity (Cauchy-inv) holds without the time integral and is known as Cauchy invariances (that is why we call (Cauchy-inv) the compressible Cauchy invariances). The Cauchy invariances can be thought as the 3D analogue of the fact that in 2D the vorticity is transported by the flow. (Cauchy-inv) is the generalization to compressible fluids.

(See [10] for a proof.)

Sketch of the proof of the theorem. Our strategy is to apply $\partial_t^k \bar{D}^{3-k}$ to (CFBEE-La) and contract with $\partial_t^k \bar{D}^{3-k} v$, $1 \leq k \leq 3$, since now $s = 3$. (It is more convenient to start with (CFBEE-La) rather than (CFBEE-La') and replace $q(R)$ (via (CFBEE-Le)) only later on.) As discussed, the norms

$$\|\partial_t^k \bar{D}^{3-k} v\|_0, \quad \|\partial_t^k \bar{D}^{3-k} R\|_0, \quad \text{and} \|\partial_t^{k-1} v \cdot N\|_{3-k+1,\partial}$$

will be controlled by this technique. The first two norms do not give useful control when $k = 1, 2$, but for $k = 3$ they give $\|\partial_t^3 v\|_0$, $\|\partial_t^3 R\|_0$, which are two of the quantities we want to control.

Thus, let us assume for now that we have derived the estimate

$$\|\partial_t^3 v\|_0 + \|\partial_t^3 R\|_0 + \|\partial_t^2 v \cdot N\|_{1,\partial} + \|\partial_t v \cdot N\|_{2,\partial} + \|v \cdot N\|_{2,\partial} \leq P(N(0)) + P(N(t)) \int_0^t P(N(\tau)) d\tau$$

(CFBEE-L-est)

where P denotes a generic (i.e., possibly varying from line to line) continues function of its argument and

$$N(t) = \|v\|_3 + \|\partial_t v\|_2 + \|\partial_t^2 v\|_1 + \|\partial_t^3 v\|_0 + \|R\|_3 + \|\partial_t R\|_2 + \|\partial_t^2 R\|_1 + \|\partial_t^3 R\|_0$$

and show how to obtain the estimate of the theorem. Then, we will illustrate how to obtain (CFBEE-L-est). (Note that the last term in (CFBEE-L-est) does not correspond to $k = 1$. For technical reasons, the $k = 1$ case has to be treated differently and gives only an estimate for $\|v \cdot N\|_{2.5,\partial}$, which suffices for estimating $\|v\|_3$; see (div-curl-est) below).

We will use the following elliptic estimate for a vector field X:

$$\|X\|_s \leq C(\|\mathrm{div}(X)\|_{s-1} + \|\mathrm{curl}(X)\|_{s-1} + \|X \cdot N\|_{s-\frac{1}{2},\partial} + \|X\|_0), \quad \text{(div-curl-est)}$$

for $s \geq 1$ (this estimate is well-known; see [5] for a modern proof and references therein). We will apply this estimate for $X = v, \partial_t v, \partial_t^2 v$ ($\partial_t^3 v$ is already controlled by (CFBEE-L-est)).

We first estimate $\partial_t^2 v$ in H^1, so we use (div-curl-est) with $X = \partial_t^2 v$. Thus we need to estimate $\mathrm{div}(\partial_t^2 v)$ and $\mathrm{curl}(\partial_t^2 v)$ in L^2. Differentiating (CFBEE-Lb) in time twice gives

$$R a^{\mu\alpha} \partial_\mu \partial_t^2 v_\alpha \sim \partial_t^3 R,$$

where $\sim$ indicates modulo terms that can be estimated by standard methods (Sobolev embedding, Young's inequality, interpolation, fundamental theorem of calculus, etc.)

For small time, $a^{\mu\alpha} \partial_\mu \partial_t^2 v_\alpha \approx \mathrm{div}(v)$ (recall that $a(0) = I$), so taking the L^2 norm and using (CFBEE-L-est),

$$\|\mathrm{div}(\partial_t^2 v)\|_0 \leq P(N(0)) + P(N) \int_0^t P(N).$$

Next, we estimate $\|\text{curl}(\partial_t^2 v)\|_0$. Taking ∂_t^2 of (Cauchy-inv) and using $\partial_t \eta = v$ gives

$$\epsilon^{\alpha\beta\gamma}\partial_\beta\partial_t^2 v_\mu \partial_\gamma \eta^\mu \sim \epsilon^{\alpha\beta\gamma}\partial_\beta v_\mu \partial_\gamma \partial_t v^\mu + P(\nabla\partial_t R, \nabla\partial_t v) \sim P(N(0)) + \int_0^t P(\nabla\partial_t^2 R, \nabla\partial_t^2 v).$$

For small time, $\epsilon^{\alpha\beta\gamma}\partial_\beta\partial_t^2 v_\mu \partial_\gamma \eta^\mu \approx (\text{curl}(\partial_t^2 v))^\alpha$ (recall that $\eta(0) = \text{id}$) so taking the L^2 norm and invoking (CFBEE-L-est) gives

$$\|\text{curl}(\partial_t^2 v)\|_0 \le P(N(0)) + P(N) \int_0^t P(N).$$

Using these estimates for div and curl of $\partial_t^2 v$ into (div-curl-est), invoking the estimate for $\partial_t^2 v \cdot N$ from (CFBEE-L-est) gives

$$\|\partial_t^2 v\|_1 \le P(N(0)) + P(N) \int_0^t P(N).$$

Now let us move to estimate $\partial_t^2 R$ in H^1. Taking ∂_t^2 of (CFBEE-La') gives

$$\begin{aligned} q'(R) a^{\mu\alpha}\partial_\mu \partial_t^2 R &\sim R\partial_t^3 v^\alpha + \partial_t^2 R \partial_t v^\alpha \\ &\sim R\partial_t^3 v^\alpha \end{aligned}$$

Since $a^{\mu\alpha}\partial_\mu\partial_t^2 R \approx \delta^{\mu\alpha}\partial_\mu\partial_t^2 R$ for small time, taking the L^2 norm, using (CFBEE-L-est) we find

$$\|\partial_t^2 R\|_1 \le P(N(0)) + P(N) \int_0^t P(N),$$

We now continue in this top-down fashion, estimating $\partial_t v$ in terms of $\partial_t^2 v$ and $\partial_t^2 R$ and so on. We arrive at

$$N(t) \le P(N(0)) + P(N(t)) \int_0^t P(N(\tau)) d\tau.$$

A continuity argument now produces

$$N(t) \le P(N(0)).$$

We no turn our attention to illustrate how (CFBEE-L-est) is derived. We consider $\partial_t^k \bar{D}^{3-k}$ with $k = 3$. Thus taking ∂_t^3 of (CFBEE-a) and proceeding as previously discussed produces a bound for

$$\|\partial_t^3 v\|_0 \quad \text{and} \quad \|\partial_t^3 R\|_0,$$

provided that we can bound the boundary term

$$I = \int_{\partial\Omega} \partial_t^3 (A^{\mu\alpha} N_\mu q) \partial_t^3 v_\alpha.$$

To control this term, we will use the boundary condition. Recall that we mentioned that the boundary condition has to be written in a different way, which is as follows:

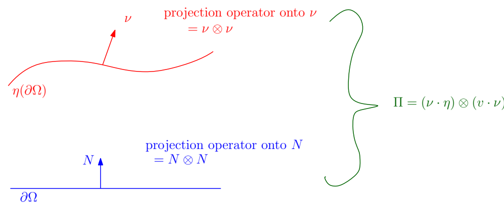
$$A^{\mu\alpha} N_\mu q = -\sigma\sqrt{|g|} g^{ij} \Pi^{\alpha\mu} \partial_{ij}^2 \eta_\mu,$$

where

$$\Pi^{\alpha\beta} = \delta^{\alpha\beta} - g^{k\ell}\partial_k\eta^\alpha\partial_\ell\eta^\beta$$

is the canonical projection from $T(\overline{\eta(\Omega)})|_{\eta(\partial\Omega)}$ onto $N(\eta(\partial\Omega))$, where $T(\overline{\eta(\Omega)})$ is the tangential bundle of $\overline{\eta(\Omega)}$ and $N(\eta(\partial\Omega))$ is the normal bundle of $\eta(\partial\Omega)$.

Remark 6.9. What comes directly out of a priori estimates are bounds for $\Pi(\partial_t^k v)$, which in general equals $\partial_t^k v\cdot N$ only at $t=0$. But for small times we can compare the two quantities.



The integral I now becomes

$$\begin{aligned} I &= -\sigma\int_{\partial\Omega}\sqrt{|g|}g^{ij}\Pi^{\alpha\beta}\partial^2_{ij}\partial_t^2 v_\alpha\partial_t^3 v_\beta - 3\sigma\int_{\partial\Omega}\partial_t(\sqrt{|g|}g^{ij}\Pi^{\alpha\beta})\partial^2_{ij}\partial_t v_\alpha\partial_t^3 v_\beta \\ &- 3\sigma\int_{\partial\Omega}\partial_t^2(\sqrt{|g|}g^{ij}\Pi^{\alpha\beta}\partial^2_{ij}v_\alpha\partial_t^3 v_\beta - \sigma\int_{\partial\Omega}\partial_t^3(\sqrt{|g|}g^{ij}\Pi^{\alpha\beta})\partial^2_{ij}\eta_\alpha\partial_t^3 v_\beta \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

I_1 produces the coercive boundary term. For, integrating by parts ∂_i, using the symmetry of $\Pi^{\alpha\beta}$ and the identity

$$\Pi^{\alpha\beta} = \Pi^{\alpha\mu}\Pi_\mu{}^\beta$$

produces

$$\begin{aligned} I_1 &\sim \frac{1}{2}\sigma\partial_t\int_{\partial\Omega}\sqrt{|g|}g^{ij}\Pi^{\alpha\mu}\partial_j\partial_t^2 v_\alpha\Pi_\mu^\beta\partial_i\partial_t^2 v_\beta \\ &\sim \frac{1}{2}\sigma\partial_t\|\Pi\bar{D}\partial_t^2 v\|^2_{0,\partial} \sim \frac{1}{2}\sigma\partial_t\|\Pi\partial_t^2 v\|^2_{1,\partial} \\ &\sim \frac{1}{2}\sigma\partial_t\|\partial_t^2 v\cdot N\|^2_{1,\partial}. \end{aligned}$$

(Note that we get the correct sign).

Note that the integrals I_2, I_3, and I_4 are all problematic not only because of the term $\partial_t^3 v$, which we can only bound in L^2 of the interior (Recall our previous discussion of top order terms on the boundary), but because of the other terms as well. For instance, in I_4 we have

$$\partial_t^3 g \sim \partial_t^3\bar{D}\eta \sim \partial_t^2\bar{v}.$$

Since $\partial_t^2 v\in H^1(\Omega)$, $\partial_t^2\bar{D}v\in L^2(\Omega)$ so we cannot bound this term on $\partial\Omega$. Moreover, similar bad terms appear among the terms we omitted in I_1 above.

The control of all such terms follows from a very delicate analysis of the boundary integrals that involves a large variety of geometric identities that are combined with the equations leading to remarkable cancellations. For example,after a long series of applications of such ideas, we are able to control all terms coming from I_1 (omitted above) except one, call it B_1. Similarly, after a long process we can control all terms coming from I_4, except one, call it B_4. B_1 and B_4 exactly cancel each other.

See [15] for details of the proof. □

Remark 6.10. Several of the complications discussed above do not arise when $\sigma = 0$ because the troubling boundary terms are simply not present in this case. (Although other difficulties are present when $\sigma = 0$.)

7. Relativistic fluids

By a relativistic fluid we mean a fluid in a regime where the laws of relativity cannot be neglected. The field of relativistic hydrodynamics or relativistic fluid dynamics is concerned with the study of relativistic fluids. Relativistic fluid dynamics is an essential tool in high-energy nuclear physics, cosmology and astrophysics [40].

One models relativistic fluids by considering Einstein's equations (EE) coupled to a fluid source, i.e., to an energy momentum tensor (a.k.a stress-energy tensor) of a fluid:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta}, \quad \text{(EE)}$$

where $T_{\alpha\beta}$ is the fluid's energy-momentum tensor. The choice of $T_\alpha \beta$ depends on the type of fluid we want to study (we will see examples). As a consequence of the Bianchi identities)

$$\nabla_\alpha R_{\beta\gamma}{}^{\lambda}{}_{\mu} + \nabla_\beta R_{\gamma\alpha}{}^{\lambda}{}_{\mu} + \nabla_\gamma R_{\alpha\beta}{}^{\lambda}{}_{\mu} = 0$$

the LHS of EE is divergence-free so we necessarily have

$$\nabla_\alpha T^\alpha{}_\beta = 0 \quad \text{(div-T)}$$

i.e., $T_{\alpha\beta}$ must be divergence-free (recall that $T_{\alpha\beta}$ is symmetric).

Even though (div-T) is a consequence of (EE), from the point of view of the initial value formulation we consider the system (EE)+(div-T) in order to obtain a closed system of PDEs.

In many applications we consider solely equations (div-T) with a given Lorentzian metric (which typically solves the vacuum-EE). The motivation for this is the following. Restoring units, (EE) read

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

where G is Newton's constant and c is the value of the speed of light in vacuum (equations (EE) are written in units such that $8\pi G = 1 = c$). If, in the same units we measure G and c, the fluid variables entering in $T_{\alpha\beta}$ are not "too big" (in whatever sense we can make this statement) then, since $\frac{G}{c^4}$ is "small", we have $\frac{8\pi G}{c^4} T_{\alpha\beta} \approx 0$. Thus we can consider the vacuum EE and solve for g independently of $T_{\alpha\beta}$, which in practice means that from the point of view of $T_{\alpha\beta}$ the metric is given. The equations of motion for the fluid variables are still given by (div-T). This corresponds to a physical situation where the fluid "feels" gravity (since (div-T) involves $g_{\alpha\beta}$ that solves the vacuum EE), but gravity is not affected by the fluid (since we find g solving the vacuum EE). Note that the fluid is still relativistic since its dynamics depends on $g_{\alpha\beta}$. This situation occurs, for example, in certain types of fluid matter that forms in particle accelerators, where the fluid moves near the speed of light (hence relativistically) but the space-time curvature is in practice not affected by the fluid. In this case, as many cases of practical interest in Earth-based experiments with relativistic matter, we take $g_{\alpha\beta}$ to be the Minkowski metric.

When $g_{\alpha\beta}$ is given in (div-T), we say that we have the fluid equations in a fixed background.

We will consider two types of relativistic fluids: with and without viscosity, and both coupled to Einstein and in a fixed background.

8. The relativistic Euler equations

The relativistic Euler equations model a perfect relativistic fluid, i.e., a fluid with no viscosity. More precisely, we consider the Einstein-Euler (EEu) system given by (EE) with

$$T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + pg_{\alpha\beta} \quad \text{(Euler-tensor)}$$

where u_α is the fluid's (four-) velocity, normalized such that

$$u_\alpha u^\alpha = -1 \quad \text{(u-unit)}$$

ρ is the fluid's (energy) density, and p is the fluid's pressure. The pressure and the density are connected by an equation of state $p = p(\rho)$ as in the case of the CEE. It turns out that it is an empirical fact (see [40]) that in many situations the pressure depends not only on ρ but also on other thermodynamic variables such as entropy, enthalpy, baryon number, etc. (see [40]). for example, we can have $p = p(\rho, s)$, where s is the entropy. From the laws of thermodynamics and the equation of state, two thermodynamic quantities (say, ρ and s) determine all others. The choice of which two thermodynamic quantities are chosen as independent is a matter of convenience, although certain choices are preferable depending on the problem.

If we have an equation of state where p depends on two thermodynamic quantities, then we need to introduce a new equation of motion in order to obtain a closed system of PDEs. Many times it is convenient to choose as independent thermodynamic variables the density ρ and the baryon number n, $p(\rho, n)$. The equation we postulate for n is (see [40])

$$\nabla_\mu(nu^\mu) = 0. \quad \text{(baryon-eq)}$$

Equations (div-T) for (Euler-Tensor) can be decomposed in the directions parallel and orthogonal to u_α using (n-unit). We find

$$u^\mu \nabla_\mu \rho + cp + \rho)\nabla_\mu u^\mu = 0, \quad \text{(REEa)}$$

$$(p+\rho)u^\mu \nabla_\mu u_\alpha + \Pi_\alpha^\mu \nabla_\mu p = 0, \quad \text{(REEb)}$$

where $\Pi_{\alpha\beta}$ is the projection on the space orthogonal to u_α, which for u_α satisfying (u-unit), is given by

$$\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta.$$

Once an equation of state is given, equations (REEa)-(REEb) + (u-unit) (+ (baryon-eq) if p depends on two thermodynamic variables) are known as the relativistic Euler equations (REE) (which can be studied coupled to Einstein or in a fixed background).

We think of (u-unit) as a constraint that is propagated by the flow. In fact, we have:

Proposition 8.1. *For solutions of the REE, $u_\alpha u^\alpha = -1$ if this condition is satisfied initially.*

To study the initial value problem, we write the EEu equations as (EE)+(REE).

Theorem 8.2. *(informal version) The initial value problem for the EEu system is locally well posed if $\rho(0) \geq$ constant > 0.*

sketch of proof. The REE can be written as a quasi-linear first order symmetric hyperbolic system (see [2]), as do the EE (see [22]). The coupled system remains symmetric hyperbolic. $\square$

Remark 8.3. The same reasoning shows that the REE in a fixed background are locally well posed

Remark 8.4. Note the assumption $\rho(0) \geq \text{const.} > 0$ (as for the classical compressible Euler equations). If $\rho(0) = 0$ is allowed, we have the free-boundary relativistic Euler equations, for which the initial value problem is largely open (see [28])

Remark 8.5. In the above theorem and also in the study of the CEE, we relied on first order symmetric hyperbolic theory to provide a quick proof of local existence and uniqueness. It should be stressed however, that writing these equations in symmetric hyperbolic form is not always the best way to study them.

It is known that the REE form shocks (see [7]). We will present a formulation of the REE tailored to the study of shock formations. Thus, it is instructive to first say a few words about the mathematical theory of shock waves.

8.1. **Shock waves.** Roughly speaking, a shock is a region in space-time where a derivative of the solution blows-up while the solution itself remains bounded. Shocks are not merely mathematical curiosities but do model real physical phenomena (see [40]).

Since the physical world does not "cease to exist" after the formation of a shock, it is important to understand how one can continue the solution past the shock. While this is well-understood in one spatial dimension (see: systems of conservation laws; Rankine-Hugoniot conditions), in more than one spatial dimension the problem remains largely open.

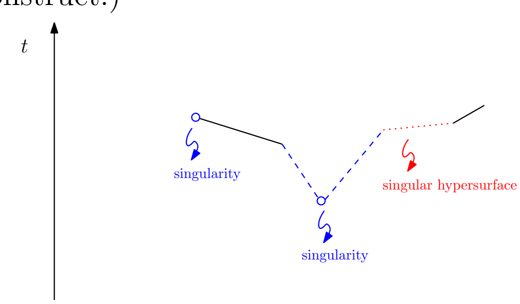
Technical note. The theory in 1d is successful in large part because it accommodates estimates in the bounded variation norm:

$$\|f\|_{BV([a,b])} = \sup_p \sum_{i=0}^{\ell_p - 1} |f(x_{i+1}) - f(x_i)|$$

$\sup_p = \sup$ over all partitions of $[a, b]$, $x_i \leq x_{i+1}$, $0 \leq i \leq \ell_p - 1$.

Such BV-estimates are in general not true in higher dimensions (see [42] for more discussion)

More precisely, we would like to continue the solution past the shock in a weak sense. For this, it is not enough to know that a shock forms, but we need a complete description of the shock profile. (Roughly, think of the shock profile as the "initial data" for the weak formulation we want to construct.)



The known constructive proofs of stable shock formation rely on the following ingredients:

(S1) The shock is driven by a Riccati-type term.

(S2) The equations admit a formulation that "hides" the Ricatti term and exhibits a "good" null-structure (null-forms).

(S3) There exist suitably adapted coordinates whose regularity theory is compatible with the formulation of the equations in (S2)

Remark 8.6. Null-forms are typically associated with problems of global existence. Let us illustrate their role in the study of shocks with the following ODE example. Consider the Ricatti ODE $\dot{y} = y^2$ which blows-up in finite time. We can ask: what kinds of perturbations do not alter the character of the blow-up of the Ricatti equation? We see, e.g., that solutions to the perturbed equation $\dot{y} = y^2 + \epsilon y$ blow-up like those of $\dot{y} = y^2$, while solutions to $\dot{y} = y^2 + \epsilon y^3$ might blow-up or exist globally depending on the sign of ϵ (taking, say, y(0)=1). The null forms play a role analogues to the perturbation $+\epsilon y$, i.e., they do not alter the character of the blow-up (so the null-forms do not exhibit the most "singular" type of non-linearities).

Next, we will present a new formulation of the REE that enjoy properties (S1), (S2), and (S3).

8.2. **A new formulation of the relativistic Euler equations.** For our new formulation we will take as independent thermodynamic variables the entropy s nad the log-enthalpy h (that is the logarithm of the enthalpy). As in the case of the CEE, we define the fluid's sound speed by

$$c^2 = \frac{\partial p}{\partial \rho},$$

which can be rewritten as $c^2 = c^2(h, s)$. We will work always under the assumption that

$$0 < c < 1$$

Definition 8.7. We define the acoustical metric $G_{\alpha\beta}$ by

$$G_{\alpha\beta} = \frac{1}{c^2} g_{\alpha\beta} + \left(\frac{1}{c^2} - 1\right) u_\alpha u_\beta.$$

Its inverse is given by

$$(G^{-1})^{\alpha\beta} = c^2 (g^{-1})^{\alpha\beta} - (1 - c^2) u^\alpha u^\beta,$$

where u_α and u^α have their indices raised and lowered with respect to the metric $g_{\alpha\beta}$, and u^α and c are the fluid's velocity and sound speed.

We continue to raise and lower indices with the metric $g_{\alpha\beta}$, but in order to avoid possible confusion we indicate the inverses of $g_{\alpha\beta}$ and $G_{\alpha\beta}$ not only by their upper indices but also by $(\cdot)^{-1}$.

One can verify that $G_{\alpha\beta}$ is in fact a Lorentzian metric with inverse $(G^{-1})^{\alpha\beta}$. Thus, all machinery of Lorentzian geometry applies to the geometry of $G_{\alpha\beta}$. In particular, the null-vectors with respect to $G_{\alpha\beta}$ form cones that are called sound cones and correspond to the propagation of sound waves, very much like the light cone for the Minkowski metric. Naturally, the geometry of $G_{\alpha\beta}$ is tied to the fluid, and is present and in general non-trivial even if $g_{\alpha\beta}$ is the Minkowski metric.

The null-forms that will be important for our new formulation will be relative to the acoustical metric.

Definition 8.8. We define the standard null-forms relative to $G_{\alpha\beta}$ as

$$\mathscr{N}^G(\partial\varphi, \partial\psi) = (G^{-1})^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \psi,$$

$$\mathscr{N}^G_{\alpha\beta}(\partial\varphi, \partial\psi) = \partial_\alpha \varphi \partial_\beta \psi - \partial_\beta \varphi \partial_\alpha \psi$$

In order to state the new formulation, we need some further notation.

Notation 8.9.

- $\omega^\alpha \sim \partial u + \partial h$ is the (four-) vorticity .
- $S_\alpha = \partial_\alpha s$ is the entropy gradient.
- $\mathscr{C} \sim \partial^2 u$ is a modified version of the vorticity of ω^α, i.e., the velocity of the vorticity.
- $D \sim \partial^2 s$ is a modified version of $\nabla_\alpha s^\alpha$.
- $\mathscr{N}(T_1, \cdots, T_m)$ denotes linear combinations of the standard null forms relative to g.
- $\mathscr{L}(T_1, \cdots, T_m)$ denotes linear combinations of terms that are at most linear in $\partial T_1, \cdots, \partial T_m$.
- $\square_G$ is the covariant wave operator relative to the acoustical metric.

The new formulation of the REE can be stated as follows.

Theorem 8.10. *(informal version) Solutions to the REE also satisfy the following system:*
wave-equations

$$\square_G h \sim D + \mathscr{N}(\partial h, \partial u) + \mathscr{L}(\partial h),$$
$$\square_G u^\alpha \sim \mathscr{C}^\alpha + \mathscr{N}(\partial h, \partial u) + \mathscr{L}(\partial h, \partial u)$$

transport equations

$$u^\alpha \partial_\alpha s = 0$$
$$u^\lambda \partial_\lambda S^\alpha \sim \mathscr{L}(\partial u)$$
$$u^\lambda \partial_\lambda \omega^\alpha \sim \mathscr{L}(\partial h, \partial u)$$

div-curl transport system

$$u^\lambda \partial_\lambda D \sim \mathscr{C} + \mathscr{N}(\partial S, \partial h, \partial u) + \mathscr{L}(\partial h, \partial u)$$
$$\mathrm{vort}^\alpha(S) = 0$$
$$u^\lambda \partial_\lambda \mathscr{C}^\alpha \sim \mathscr{C} + D + \mathscr{N}(\partial S, \partial\omega, \partial h, \partial u) + \mathscr{L}(\partial S, \partial\omega, \partial h, \partial u)$$
$$\partial_\alpha \omega^\alpha \sim \mathscr{L}(\partial h).$$

Moreover, using these equations, we can prove a local well-posedness result for the REE in which S_α and ω^α gain one extra derivative, i.e., we obtain $u^\alpha, h \in H^N$, $s \in H^{N+1}$, $\omega^\alpha \in H^N$ (provided that such regularity holds initially).

The important point to stress is that the formulation of the REE presented in the theorem enjoys properties (S1), (S2), and (S3). (The extra regularity of s and ω^α, in particular is required for (S3).)

With this theorem we have not (yet!) solved the problem of describing the shock profile. Rather, we provide the basic framework to attack this problem. (Recall from Jared's lectures that in problems of this nature just finding the correct set-up presents a significant challenge; such a set-up is given in the above theorem.

Let us comment on the reason to introduce the modified variables $\mathscr{C}^\alpha$ and D. For our framework, we need to be able to derive good estimates for $\mathrm{vort}^\alpha(\omega)$ (the vorticity of ω^α) and $\nabla_\alpha S^\alpha$ (the divergence of S^α), but those quantities directly do not satisfy good evolution equations. Modifications of those quantities (i.e., $\mathscr{C}^\alpha$ and D), however, do not satisfy good equations. Information about $\mathrm{vort}^\alpha(\omega)$ and $\nabla_\alpha S^\alpha$ can then later be obtained from $\mathscr{C}^\alpha$ and D.

See [16] for a proof of the above theorem.

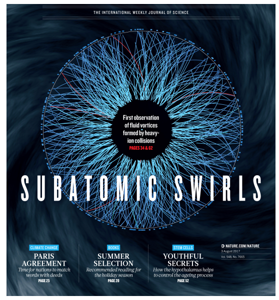


Figure 1. Cover of Nature reporting the "polarization of lambda hadrons," indicating that the vorticity of QGP is extremely high.

9. Relativistic viscous fluids

So far, we have only discussed fluids without viscosity. The classical theory of viscous fluids is described by the Navier-Stokes equations, which we will not address here. For relativistic fluids, we can ask if there is a need to consider fluids with viscosity, given the great success of the REE in applications (see [40]).

Therefore let us start highlighting how there are important physical applications where relativistic viscous fluids (RVF) are important.

The quark-gluen-plasma (QGP) is an exotic type of matter that forms in collisions of heavy-ions, such as those performed at CERN's Large Hadron Collider (LHC) or at Brookhaven National Lab's Relativistic Heavy Ion Collider (RHIC). The discovery of the QGP was named by the American Physical society one of the most important scientific findings in physics in the last decade. And it continues to be a source of scientific breakthroughs. For example, recently it has been discovered that the QGP is the most vertical fluid known to date, a finding that featured on the cover of the journal Nature in August 2017.

So there is no doubt that the QGP is a very important physical system. Here, what is important to know is that theoretical predictions to the QGP match experimental data only if viscosity is included. This is illustrated in the following graph.

Another important situation where RVF are important is in the study of neutron stars. The direct detection of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016 made gravitational-wave astronomy into a reality. More recently, newly reached precision in locating the direction in the sky of the sources of gravitational waves allows researches to study events producing simultaneously gravitational waves and emissions of light, of which neutron stars mergers are primary examples as recently discovered by LIGO (Oct 2017); see illustration below.

There has been increasing awareness of the importance of viscosity in the dynamics of neutron stars, and recent state-of-the-art numerical simulations strongly suggest that viscosity cannot be neglected in neutron star mergers [1])

In conclusion, one has two of the most cutting-edge experimental apparatus in modern science (LHC and LIGO) producing data that requires RUF for its explanation.

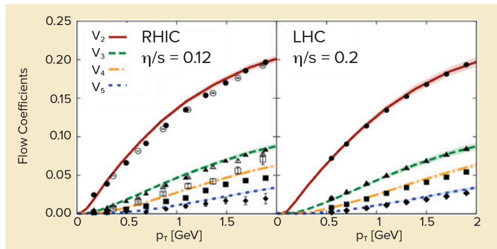


FIGURE 2. (Fourier coefficients of the angular distribution of hadrons vs. transverse energy. source: 2015 Long Range Plan for Nuclear Sciences, DOE & NSF.)

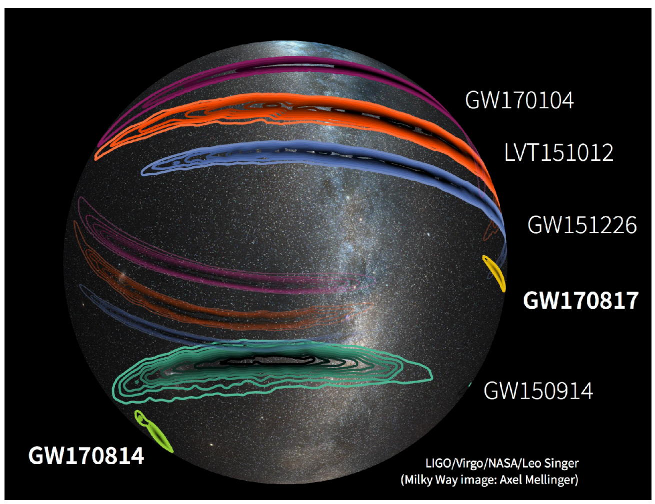


FIGURE 3. Illustration of Ligo's increased precision. The gravitational wave detections highlighted (GW170817,GW170814) can be placed in a small portion of the sky as compared to previous detections (also showed in the picture). (Source: LIGO GW170817 Press Release)

Contrasting with these extraordinary advances on the experimental side, the theory of RVF is largely underdeveloped, as we will now see.

9.1. **Equations of motion for RVF.** As seen, to study a particular matter model, we need to identify an energy momentum tensor. Unlike the case of a prefect fluid, it is not known what the energy-momentum tensor of a RVF is. This is because the physical arguments used to motivate the definition of $T_{\alpha\beta}$ for a perfect fluid no longer apply in the case of fluids with viscosity (see [45]). We can, of course, always postulate a particular $T_{\alpha\beta}$, but one needs some physical principle to guide our choices. As we will see, the most "natural" (in the sense that they are more or less straightforward generalizations of the classical Navier-Stokes equations to relativity) choices lead to several pathologies. Despite continuing efforts of the community we still lack a theory of RVF that meets several important physical criteria.

It turns out that it is extremely difficult to construct theories of RVF that are compatible with the principles of relativity. Some properties that we take for granted in other matter models, such as causality (i.e., the property that the equations of motion enjoy the finite-speed propagation property/domain of dependence, with the speed of propagation at most the speed of light) are hard to achieve for RVF. In other words, since dissipation is a phenomenon linked to viscosity, we are saying that is hard to model dissapative phenomena within relativity theory. At this point, it is worth taking a step back and list the properties we would like a theory of RVF to have.

9.2. **Requirements for a theory of RV.** We make the following requirements for theories of RVF:

(RVF-I) The equations of motion are locally well-posed, both

- (RVF-Ia) in Minkowski background, and
- (RVF-IIb) when coupled to Einstein's equations

We remark that going from (RVF-Ia) to (RVF-IIb) is not trivial. E.g., many theories of RVF involve a $T_{\alpha\beta}$ that contains first covariant derivatives of u_α, so (div-T) involves two derivatives of $g_{\alpha\beta}$, the same as (EE).

(This comes from properties of viscosity; think of the Navier-Stokes equations which are second order in u^α as compared to the Euler equations that are first order).

RVF-II The equations of motion are causal.

This requirement, which is obtained "for free" in most matter models, is included here because it is difficult to ensure its validity when viscosity is present.

RVF-III The equation of motion in Minkowski background (i.e., for $g_{\alpha\beta}$ = Minkowski) are linearly stable about thermodynamic equilibria, which in practice we take as a constant state.

A thermodynamic equilibrium is a solution to the equations of motion for which the following holds. The "dissipative function," i.e., the terms in the equations of motion that correspond to the contribution of viscosity, vanish, so that the equations reduce to those of a perfect fluid. Thinking of the classical incommpressible Navier-Stokes equations as an example,

$$\partial_t u + \nabla_u u + \nabla p - v \triangle u = 0,$$

where v is the viscosity, a thermodynamic equilibrium is a solution for which $\triangle u = 0$. Note that we are talking about thermodynamic equilibrium and not about dynamic equilibrium. A perfect fluid, for example, is in thermodynamic equilibrium (since it has no dissipation), even though its dynamics can be quite complex.

We now explain what is meant by linear stability (about thermodynamic equilibria). Suppose our equations of motion for the fluid in Minkowski background read, symbolically,

$$P(\varphi) = 0,$$

where φ represents the fluid variables. Let Ψ be a thermodynamic equilibrium solution, and linearize the equations about Ψ, obtaining a linear equation with Ψ-dependent coefficients:

$$L_\Psi(\varphi) = 0.$$

Now we solve this equations by setting $\varphi(t,x) = e^{i(\omega t + k\cdot x)}\varphi_0$, φ_0 a constant vector, and plugging into $L_\Psi(\varphi) = 0$, producing an algebraic equation for (ω, k). We find the roots $\omega = \omega(k)$. We call $P(\varphi) = 0$ linearly stable (about thermodynamic equilibria) if, for all such Ψ, the roots $\omega(k)$ have positive imaginary part (for all k).

The underlying physical principle in this definition is the idea that if we perturb a system with viscosity out of an equilibrium configuration, then the system should return to equilibrium due to the effects of dissipation. Thus, the perturbation has to decay in time, and this will be case if $Im(\omega(k)) > 0$. In practice, determining Ψ can be very complicated, and one restricts to the cases where Ψ is constant.

Remark 9.1. We will henceforth refer to linear stability about thermodynamic equilibrium simply as stability.

RVF-IV The equations of motion have to be derivable from a more fundamental microscope theory (in certain approximations).

We know that the fluid equations are only an approximation (a "continuum limit") of a more fundamental microscopic theory, typically from kinetic theory governed by Boltzmann equations. For macroscopic theories that have been well-tested, their derivation from microscopic theory might be considered a theoretical open problem whose outstanding lack of solution is unlikely to shake our trust in the theory. For RVF, however, one is trying to introduce a new theory (set of equations for which we have not much guidance on how to proceed. In this case, obtaining the equations from a microscopic theorey is an important ingredient to "keep us honest", putting a potentially speculative new theory into a more firm basis. In fact, as a rule of thumb, one should be very suspicious of theories that cannot be derived from a more fundamental microscopic theory.

RVF-V The theory describes relevant physics.

What counts as describing relevant physics is open for discussion. Therefore, we will limit ourselves to mentioning whether or not applications of a given theory have been developed.

9.3. **Brief review of theories of RVF.** We will now provide a brief review of the literature dealing with RVF. Our list is far from complete, aiming only to illustrate how researchers have struggled to construct theories satisfying properties (RVF-I)-(RVF-V). For each theory below, we indicate their standing regarding each one of such properties. We refer to [40], chapter 6, and [4] for a more thorough discussion. Naturally the results discussed below hold under suitable hypotheses that are stated in the given references. When we say that no result is known, we mean modulo trivial results (e.g., when the equations reduce to ODEs).

Notation 9.2. We henceforth denote by $t_{\alpha\beta}$ the energy-momentum tensor of a perfect fluid. We continue to use the same notation employed for perfect fluids, with u_α satisfying (u-unit).

9.3.1. *Eckart theory.* (1940, see [19]).

Eckart proposed the following energy momentum tensor for a RVF:

$$T_{\alpha\beta} = t_{\alpha\beta} - \eta \Pi^\mu_\alpha \Pi^\nu_\beta (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}) - \zeta \nabla_\lambda u^\lambda \Pi_{\alpha\beta} - k(g_\alpha u_\beta + q_\beta u_\alpha),$$

where η, ζ, and k are the coefficients of shear viscosity, bulk viscosity, and heat conduction, respectively; they are known functions of the thermodynamic variables whose form depends on the nature of the fluid. q_α is known as the fluid's heath flow. The physical interpretation of η and ζ is similar to their classical counterparts (see [40]). Readers unfamiliar with the meaning of k and q_α can set $k = 0$ without affecting the ensuing

RVF-I - There is no known local well-posedness result for Eckart's theory

RVF-II - Eckart's theory is not causal (see [40]).

RVF-III - Eckart's theory is not stable (see [40]).

RVF-IV - Eckart's theory is derivable from kinetic theory (see [26])

Remark 9.3. Kinetic theory is a well defined theory compatible with relativity. The fact that an acousal theory such as Eckart's can be derived from microscopic theory shows the limitations of the fluid approximation to kinetic theory. To derive a fluid theory from microscopic theory, one expands solutions to, say, Boltzmann's equation, in a small parameter and about an equilibrium solution, truncating the expansion at some point. Even though Boltzmann's equations are causal and locally well-posed (see [6]), there is no reason why the truncated theory should share these properties.

RVF-V - Despite all shortcomings listed above, Eckart's theory has been used to gain intuition in physical applications (see [27]; in fact, this application uses Landau's theory, which is close to the Eckart theory).

9.3.2. *Landau theory.* (1950's, see [33])
Landau's theory is very similar to Eckart's, and the same conclusions hold here.

9.3.3. *Lichnerowicz theory.* (1955, see [34])
Lichnerowicz introduced the following energy-momentum tensor:

$$T_{\alpha\beta} = t_{\alpha\beta} - \eta\Pi^{\mu}_{\alpha}\Pi^{\nu}_{\beta}(\nabla_{\mu}c_{\nu} + \nabla_{\nu}c_{\mu} - \frac{2}{3}\nabla_{\lambda}c^{\lambda}g_{\mu v}) - \zeta\nabla_{\lambda}c^{\lambda}\Pi_{\alpha\beta} - k(q_{\alpha}c_{\beta} + q_{\beta}c_{\alpha}),$$

where $c_\alpha = \frac{p+\rho}{n}$ and the other quantities are as above.

RVF-I - From irrotational fluids (i.e., fluids with no vorticity) both (RVF-Ia) and (RVF-Ib) hold (see [9]; see also [8]). It is not known whether (RVF-I) holds for rotation fluids.

RVF-II - Causality holds for irrotational fluids and it is not known whether it does for rotational fluids. (see [9]).

RVF-III - It is not known whether Lichnerowicz's theory is stable.

RVF-IV - It is not known whether Lichnerowicz's theory can be derived from a microscopic theory.

RVF-V - Lichnerowicz's theory has been applied to cosmology (see [14]).

9.3.4. *Mueller-Israel-Stewart theory.* (1970's see [38], [30], [31], and [40])
The Mueller-Israel-Stewart (MIS) theory introduces:

$$T_{\alpha\beta} = t_{\alpha\beta} + \Pi_{\alpha\beta} + \Pi\pi_{\alpha\beta} + (Q_{\alpha}u_{\beta} + Q_{\beta}u_{\alpha}).$$

The symmetric two tensor $\Pi_{\alpha\beta}$, the scalar Π, and the one-form Q_α model the dissipative effects in the fluid. In the MIS theory these fields are introduced as new variables satisfying extra equations of motion. Their equations of motion are chosen in such a way that entropy production is always non-negative. It is important to stress that these are new equations of motion on the same footing as (div-T). The coefficients η, ζ, and k are absorbed in the definition of $\Pi_{\alpha\beta}$, Π, and Q_α, which, in turn, contain further parameters.

RVF-I - There is no known (local well-posedness result for the MIS theory.

RVF-II - The linearization of the MIS equations about thermodynamic equilibria is causal (see [40]).

RVF-III - The linearization of MIS equations about thermodynamic equilibria is stable (see [40]).

RVF-IV - The MIS theory can be derived from kinetic theory (see [26]).

RVF-V - The MIS theory is currently the most widely used theory in the study of RVF, and it has been instrumental in the construction of models that provide us with great insight into the physics of RVF. For example, the above plot "Fourier coefficients..." that shows great agreement between theory and experiment relies on the MIS theory for the theoretical predictions.

9.3.5. *BRSSS theory.* (2008, see [3])

The BRSSS theory takes a different point of view as compared to the MIS theory but arrives at very similar equations. All that was said about the MIS theory applies to the BRSSS theory as well. (More precisely, these conclusions hold for what is known as the resumed BRSSS theory; see [3].

Remark 9.4. Given the success of the MIS and BRSSS theories in connecting theory with experiments, one could potentially contend that these theories settle the question of how to correctly model RVF, and that points RVF-I,II, and III would be technical open problems of interest to mathematicians but with no direct impact on physical applications. Therefore, while acknowledging the great deal of progress brought about by the MIS and BRSSS theories, some remarks about their current limitations are in order.

The matching of these theories with experimental data does not rely on fitting some parameters that enter in the model. While this is part of the normal way of doing science, it is not known whether one can describe a large variety of experimental results using the same parameter values, i.e., those values that have already been fixed by some of the data. In particular, very little is known about the applicability of the MIS and BRSSS theories to problems in astrophysics where gravity becomes relevant (so that coupling to Einstein's equations is necessary). (see [40] and references therein.)

Moreover, we should not dismiss properties RVF-I,II, and III, even if a given theory is in agreement with the data. It would be hard to make sense of a candidate for a relativistic theory if, say, it violates causality. (See [4] for further discussion.)

9.3.6. *Freistuhler and Temple(FT) theory.* (2014, see [23],[24], [25])

These authors introduce an energy-momentum tensor for RVF with several good properties (see the above references for the expression of the energy momentum tensor).

<u>RVF-I</u>

- (RVF-Ia) - The FT theory is locally well posed in Minkowski background.
- (RVF-Ib) - It is not known whether FT theory is locally well-posed when coupled to Einstein's equations

<u>RVF-II</u> - The FT theory is causal.

<u>RVF-III</u> - The authors obtained a partial stability result, as follows. They showed their theory to be stable in the fluid's rest frame, i.e. in coordinates where the fluid's velocity reads $(1, 0, 0, 0)$. This is an important first step to test the stability of a given theory. However it is known that stability in the rest frame does not imply stability in general (e.g., Landar's theory, that we saw to be unstable, happens to be stable in the rest frame, see [29]).

<u>RVF-IV</u> - It is not known whether FT theory can be derived from a microscopic theory.

<u>RVF-V</u> - To the best of our knowledge, so far no applications of the FT theory have been developed.

9.3.7. *Divergence-type theories.* (1970's, see [40]).

A large class of fluid theories can be constructed from a formalism known as divergence-type. While this formalism per se does not guarantee any of the properties RVF-I to V, it has been successfully applied to the construction of theories that are causal near equilibrium. See [40] for more details.

This brief literature review illustrates how difficult it is to construct theories of RVF that satisfy all properties. RVF-I to V. With this in mind, we will now present an energy-momentum tensor for which all properties RVF-I to V are satisfied.

9.4. **A new conformal tensor.** Consider the following energy-momentum tensor:

$$\begin{aligned} T_{\alpha\beta} =& \frac{4}{3}u_\alpha u_\beta \rho + \frac{1}{3}\rho g_{\alpha\beta} - \eta \Pi^\mu_\alpha \Pi^\nu_\beta (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}) + \frac{1}{3}\chi \nabla_\mu u^\mu \Pi_{\alpha\beta} \\ &+ \lambda (u_\alpha u^\mu \nabla_\mu u_\beta + u_\beta u^\mu \nabla_\mu u_\alpha) + \chi \nabla_\mu u^\mu u_\alpha u_\beta + \frac{\lambda}{4\rho}(u_\alpha \Pi^\mu_\beta \nabla_\mu \rho + u_\beta \Pi^\mu_\alpha \nabla_\mu \rho) \\ &+ \frac{3\chi}{4\rho} u_\alpha u_\beta u^\mu \nabla_\mu \rho + \frac{\chi}{4\rho}\Pi_{\alpha\beta} u^\beta \nabla_\mu \rho, \quad \text{(CT)} \end{aligned}$$

where $\eta = \eta(\rho)$ is the coefficient of shear viscosity, $\chi = a_1\eta$, $\lambda = a_2\eta$, a_1, a_2 constants, and, as before, $\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, and u_α and ρ are the velocity and (energy) density of fluid.

We call (CT) a conformal tensor, meaning that (div-T) remains invariant under conformal transformations of the metric. Fluids whose energy-momentum tensor satisfy this property are called conformal fluids. Such fluids are important because the QGP at very high temperatures can be modeled as a confromal fluid. For conformal fluids, $p(\rho) = \frac{1}{3}\rho$ (which we have already substituted in (CT)) and $\eta(\rho)$ is proportional to $\rho^{\frac{3}{4}}$.

All properties RVF-I to V are satisfied for the energy momentum tensor (CF). This is established in [4], to which the reader is also referred for a discussion on how we can motivate (CT). Here, we will restrict ourselves to to state a theorem about RVF-I and II, and to show some applications (thus illustrating RVF-V).

([4] was written primarily for an audience of physicists . Mathematical details regarding [4] can be found in [10]).

We now discuss the regularity class where local well-posedness holds. We work in Gevrey spaces, a class of function often used in fluid dynamics.

Definition 9.5. Let $\Omega \subseteq \mathbb{R}^4$ be a domain. A map $f : \Omega \to \mathbb{R}$ is called s-Gevrey regular if it is smooth and for every component subset K of Ω there exists a constant $C > 0$ such that for all multi-indices α and for all $x \in K$,

$$|D^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s. \quad \text{(G-est)}$$

The space of all s-Gevrey regular functions is Ω is denoted $G^{(s)}(\Omega)$.

Remark 9.6. This definition generalizes to tensors and to manifolds.

When $s = 1$, (G-est) is the known Cauchy estimate and we see that $G^{(1)}$ is the space of analytic functions. $G^{(s)}$, however, is strictly larger than $G^{(1)}$ for $s > 1$. In fact,

$$G^{(1)} \subset G^{(s)} \subset C^\infty \quad (s > 1)$$

where these inclusions are proper.

Contrary to analytic functions, Gevrey spaces admit compactly supported functions (for $s > 1$) which are of course an important tool in analysis.

We will refer (EE) with $T_{\alpha\beta}$ given by (CT) and u_α satisfying (u-unit) as the viscous Einstein conformal fluid (VECF) system.

An initial set for the VECF-equations consists of a three-manifold Σ endowed with a Riemannian metric g_0, a symmetric two-tensor k, two vector fields v_0 and v_1, in Σ (thought of as the velocity and its time derivative at time zero), and two functions ρ_0 and ρ_1 (though of as the density and its time derivative at time zero), such that the constraint equations are satisfied. Note that only tangential directions of u and of its "time derivative" are given as initial data in light of (u-unit).

Theorem 9.7. *Let $I = (\Sigma, g_0, k, \rho_0, \rho_1, v_0, v_1)$ be an initial-data set for the VECF system. Assume that Σ is compact without boundary and that $\rho_0 > 0$. In the definition of χ and λ, assume that $a_1 = 4$ and $a_2 \leq 4$, and suppose that $\eta : (0, \infty) \to (0, \infty)$ is analytic. Finally, suppose that the initial data is in $C^{(s)}(\Sigma)$, $1 < s < \frac{17}{16}$. Then:*

(1) There exists a globally hyperbolic development M of I.

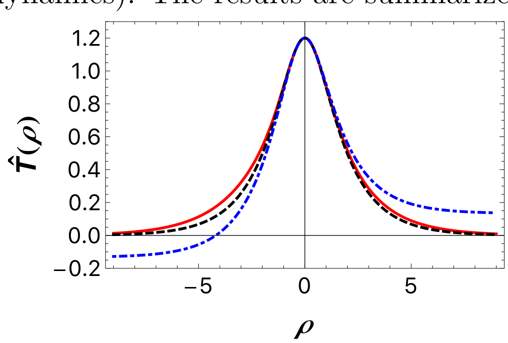
(2) M is causal, in the following sense, Let (g, ρ, u) be a solution to the VECF system provided by the globally hyperbolic development M. For any $x \in M$ in the future of $i(\Sigma)$, $(g(x), \rho(x), u(x))$ depends only on $I|_{i(\Sigma) \cap J^-(x)}$ where $J^-(x)$ is the casual part of x and $i : \Sigma \to M$ is the embedding associated with the globally hyperbolic development M.

The assumptions on a_1 and a_2 are technical. Σ is assumed compact fort simplicity, as otherwise asymptotic conditions need to be imposed. $\rho_0 > 0$ is necessarily for (CT) to be well-defined. While (CT) is tailored for conformal fluids, in which case η is proportional to $\rho^{\frac{3}{4}}$, our result is more general, requiring only η to be an analytic function of ρ. But note that η is not allowed to vanish (in particular, we cannot deduce a result for the REE as a particular case of this theorem).

We employed Gevrey spaces because, for (CT), equation (div-T) can be written in a way that constitutes a weakly hyperbolic system. For such systems , it is extremely challenging to close estimates in Sobolev spaces, but many times one can close the estimates in Gevrey spaces (in fact, there are examples of weakly hyperbolic equations that are not well-posed in Sobolev spaces but are well-posed in Gevrey spaces). While it remains an important question whether the EVCF system admits a local existence and uniqueness result in Sobolev spaces, it is important to stress that the causality condition of the theorem is a structural feature of the equations and will automatically carry over to larger function spaces where existence and uniqueness can be established. The techniques to deal with weakly hyperbolic systems go back to the seminal work of Jean Leray. See [10] for background (including the definition of weakly hyperbolic) and references, and also for a proof of the theorem.

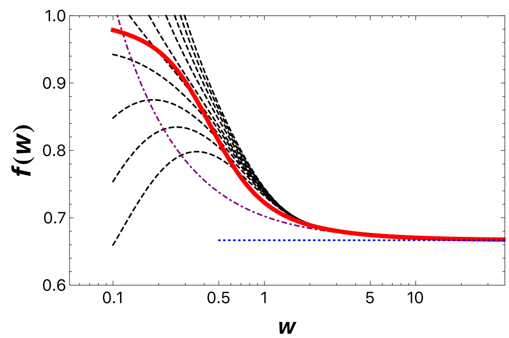
To finalize our discussion of (CT), we briefly mention two applications.

The first application is the Gubser flow. This is a simple model of heavy-ion collisions often used in the study of the QGP. It can be applied to any conformal fluid, but the details of the dynamics depend on the form of the energy momentum tensor. In our case, we investigate the temperature $\hat{T}$ as a function of a natural parameter of the problem called the de Sitter time ρ (not to be confused with the density ρ; we note that $\hat{T}$ is obtained from the density ρ from the laws of thermodynamics). The results are summarized in the following graph:



The red solid curve corresponds to (CT) and the black dotted one to the perfect fluid case, i.e., to (Euler-tensor) (with $p(\rho) = \frac{1}{3}\rho$). As we expect from physical intuition, dissipation due to the presence of viscosity increases the system temperature as compared to a perfect fluid. The dotted blue curve corresponds to Landau's theory. We included it here to illustrate the sort of pathologies that can happen with a non-causal and unstable theory. In this case, the temperature (measured in GeV and normalized by a reference T_0) becomes negative.

The second application is the Bjorken flow. This flow is in fact a particular case of the Gubsev flow, but its simpler form allows us to say more. The picture below illustrates solutions to the equations of motion for the Bjorken flow using (CT). Several initial conditions are depicted, and the corresponding solutions all converge to the thermodynamic equilibrium solution (blue dotted line), as it should be. Before doing so, however, they clump together about a distinguished solution given by the red curve. Physicists refer to phenomena of this type as the presence of a hydrodynamic attractor for RVF. The purple dotted line corresponds to Landau's theory.



We refer to [4] for further discussions of these applications.

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