VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 2

1. Problems

Unless stated otherwise, the notation below is as in class.

Problem 1. Show that Laplace's equation is rotationally invariant, i.e., if u solves $\Delta u = 0$ and we define

$$\tilde{u}(x) = u(Mx),$$

where M is an orthogonal matrix, then $\Delta \tilde{u} = 0$.

Problem 2. Prove the following fact that we used in the construction of solutions to Poisson's equation: let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, then

$$\lim_{r \to 0^+} \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS = f(x).$$

Hint: Consider the difference $f(x) - \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS$ and use $\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS = 1$.
Remark: The result is valid under weaker assumptions; in fact, it holds for a.e. x_0 if f is assumed to be locally integrable (this is sometimes known as the Lebesgue differentiation theorem).

Problem 3. In class, we constructed solutions to Poisson's equation in $\mathbb{R}^n$ for $n \geq 3$. Carry out the construction in the case $n = 2$. You do *not* have to do all the steps. Rather, follow what was done in class and point out what changes in $n = 2$. This boils down to slightly modifying some of the estimates for the fundamental solution.

Problem 4. Let u be a non-trivial harmonic function in $\mathbb{R}^n$. Can u have compact support?
Hint: mean value theorem.

Problem 5. Prove the converse of the mean value theorem. I.e., let $u \in C^2(\Omega)$ be such that

$$u(x) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \, dS$$

for each $B_r(x) \subset\subset \Omega$. Show that $\Delta u = 0$ in Ω.
Hint: Assume that $\Delta u(x) \neq 0$ for some $x \in \Omega$. Use the functions $f(r)$, $f'(r)$ used in the proof of the mean value to derive a contradiction.

2. Solutions

Solution 1. Write $y = Mx$. The chain rule gives

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \\ &= M_{ji} \frac{\partial}{\partial y^j}, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial (x^i)^2} &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\\ &= \left(M_{ji}\frac{\partial}{\partial y^j}\right)\left(M_{\ell i}\frac{\partial}{\partial y^\ell}\right)\\ &= M_{ji}M_{\ell i}\frac{\partial^2}{\partial y^j \partial y^\ell},\end{aligned}$$

where there is no sum over i above. Summing over i:

$$\begin{aligned}\Delta_x &= \sum_i \frac{\partial^2}{\partial (x^i)^2}\\ &= \sum_i M_{ji}M_{\ell i}\frac{\partial^2}{\partial y^j \partial y^\ell}\\ &= \delta^j_\ell \frac{\partial^2}{\partial y^j \partial y^\ell}\\ &= \sum_j \frac{\partial^2}{\partial (y^j)^2}\\ &= \Delta_y,\end{aligned}$$

where we used that $MM^T = I$, i.e.,

$$\sum_i M_{ji}M_{\ell i} = \delta_{j\ell}.$$

Solution 2. We have to prove that given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < r < \delta$ then

$$\left|\frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f\, dS - f(x)\right| < \varepsilon.$$

Write

$$\begin{aligned}\frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\, dS(y) - f(x) &= \frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\, dS - \frac{f(x)}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} dS(y)\\ &= \frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} (f(y) - f(x))\, dS(y).\end{aligned}$$

Thus

$$\left|\frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\, dS(y) - f(x)\right| \leq \frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} |f(y) - f(x)|\, dS(y).$$

Fix $\varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. If $r < \delta$, then $|x - y| < \delta$ for all $y \in \partial B_r(x)$, thus

$$\left|\frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\, dS(y) - f(x)\right| < \frac{1}{\operatorname{vol}(\partial B_r(x))}\int_{\partial B_r(x)} \varepsilon\, dS = \varepsilon.$$

Solution 3. We use the following estimates in the $n = 2$ case:

$$\int_{B_\varepsilon(0)} |\Gamma(y)|\, dy \leq C\varepsilon^2 |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and

$$\int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, dS(y) \leq C\varepsilon |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and the rest of the proof is essentially the same.

Solution 4. No. Let u be harmonic and with compact support and fix an arbitrary $x \in \mathbb{R}^n$. By the compact support property, there exists a $r > 0$ such that $u(y) = 0$ for all $y \in \partial B_r(x)$. By the mean value formula

$$u(x) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dS(y) = 0,$$

so that $u = 0$ since x is arbitrary.

Solution 5. If u is not harmonic, there exists a $x \in \Omega$ such that $\Delta u(x) \neq 0$. By assumption, the function

$$f(r) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u\, dS$$

is constant equal to $u(x)$ on the interval $(0, R)$, where $R > 0$ is a fixed number such that $B_R(x) \subset \Omega$. Thus $f'(r) = 0$ for all $r \in (0, R)$. On the other hand, by continuity, Δu has a definite sign, say positive, on a ball $B_{r_0}(x)$ for some $r_0 > 0$, which without loss of generality we can take such that $r_0 < R$. Arguing as in the proof of the mean value theorem, we find

$$f'(r_0) = \frac{1}{n\omega_n r_0^{n-1}} \int_{B_{r_0}(x)} \Delta u(y)\, dy > 0,$$

contradicting $f'(r_0) = 0$.