VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 6

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. In class, we proved that any function in $W^{k,p}(\Omega)$ can be approximated by smooth functions up to the boundary if Ω satisfies the segment condition and $1 \leq p < \infty$. It was left as a homework to show that the proof can be reduced to the case of functions with bounded support. Prove this claim.

Problem 2. Prove the change of variables formula stated in class: Let Ω and $\mathcal{D}$ be domains in $\mathbb{R}^n$. Suppose that there exists a one-to-one and onto map $\Psi : \Omega \to \mathcal{D}$ such that $\Psi^j, (\Psi^{-1})^j \in C^k(\Omega)$, have bounded derivatives, $j = 1, \dots, n$, $k \geq 1$, and $\frac{1}{C} \leq |\det D\Psi| + |\det D\Psi^{-1}| \leq C$ for some constant $C \geq 1$. Given $u \in W^{k,p}(\mathcal{D})$, $1 \leq p < \infty$, define $\tilde{\Psi}(u) : \Omega \to \mathbb{R}$ by $\tilde{\Psi}(u)(x) = u(\Psi(x))$. Then, $\tilde{\Psi}$ transforms $W^{k,p}(\mathcal{D})$ boundedly onto $W^{k,p}(\Omega)$ and has bounded inverse.

2. Solutions

Solution 1. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$, and $|D^\alpha \psi(x)| \leq C$ for $|\alpha| \leq k$. Set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. Then $\psi_\varepsilon(x) = 1$ for $|x| \leq \frac{1}{\varepsilon}$, $\psi(x) = 0$ for $|x| \geq \frac{2}{\varepsilon}$, and $|D^\alpha \psi_\varepsilon(x)| \leq C\varepsilon^{|\alpha|} \leq C$ for $|\alpha| \leq k$ and $0 < \varepsilon \leq 1$. If $u \in W^{k,p}(\Omega)$, then $u_\varepsilon := \psi_\varepsilon u$ belongs to $W^{k,p}(\Omega)$, has bounded support, and

$$|D^\alpha u_\varepsilon| \leq C \sum_{\beta \leq \alpha} |D^\beta u D^{\alpha-\beta} \psi_\varepsilon| \leq \sum_{\beta \leq \alpha} |D^\beta u|.$$

Set $\Omega_\varepsilon := \{x \in \Omega \,|\, |x| > \frac{1}{\varepsilon}\}$. Since $u - u_\varepsilon = (1 - \psi_\varepsilon)u = 0$ for $|x| \leq \frac{1}{\varepsilon}$, we have

$$\|u - u_\varepsilon\|_{W^{k,p}(\Omega)} = \|u - u_\varepsilon\|_{W^{k,p}(\Omega_\varepsilon)} \leq \|u\|_{W^{k,p}(\Omega_\varepsilon)} + \|u_\varepsilon\|_{W^{k,p}(\Omega_\varepsilon)} \leq C\|u\|_{W^{k,p}(\Omega_\varepsilon)}$$

which goes to zero when $\varepsilon \to 0$.

Solution 2. The map $\tilde{\Psi}$ is well-defined for a.e. functions since $k \geq 1$. Let $\{u_j\}$ be a sequence of smooth functions converging to u in $W^{k,p}(\Omega)$. Let $|\alpha| \leq k$. Successive applications of the chain rule and the product rule give

$$D^\alpha \tilde{\Psi}(u_j)(x) = \sum_{\beta \leq \alpha} p_{\alpha\beta}(x) D_y^\beta u_j(y) = \sum_{\beta \leq \alpha} p_{\alpha\beta}(x) \tilde{\Psi}(D^\beta u_j)(x),$$

where $y = \Psi(x)$ and $p_{\alpha\beta}$ is a polynomial of degree $\leq |\beta|$ in derivatives of Ψ^j of order $\leq |\alpha|$, $j = 1, \dots, n$.

Let $\varphi \in C_c^\infty(\Omega)$. We have

$$(-1)^{|\alpha|} \int_\Omega \tilde{\Psi}(u_j)(x) D^\alpha \varphi(x)\, dx = \sum_{\beta \leq \alpha} \int_\Omega p_{\alpha\beta}(x) \tilde{\Psi}(D^\beta u_j)(x) \varphi(x)\, dx.$$

But

$$(-1)^{|\alpha|} \int_\Omega \tilde{\Psi}(u_j)(x) D^\alpha \varphi(x)\, dx = \int_{\mathcal{D}} \underbrace{\tilde{\Psi}(u_j)(\Psi^{-1}(y))}_{=u_j(y)} (D^\alpha \varphi)(\Psi^{-1}(y)) |\det D\Psi^{-1}(y)|\, dy$$

and

$$\sum_{\beta\leq\alpha}\int_{\Omega} p_{\alpha\beta}(x)\tilde{\Psi}(D^{\beta}u_j)(x)\varphi(x)\,dx = \sum_{\alpha\leq\beta}\int_{\mathcal{D}} p_{\alpha\beta}(\Psi^{-1}(y))\,\underbrace{\tilde{\Psi}(D^{\beta}u_j)(\Psi^{-1}(y))}_{=D^{\beta}u_j(y)}\,\varphi(\Psi^{-1}(y))|\det D\Psi^{-1}(y)|\,dy.$$

Since $D^{\beta}u_j \to u$ in L^p, we can replace u_j by u above and change variables back to get

$$(-1)^{|\alpha|}\int_{\Omega}\tilde{\Psi}(u)(x)D^{\alpha}\varphi(x)\,dx = \sum_{\beta\leq\alpha}\int_{\Omega} p_{\alpha\beta}(x)\tilde{\Psi}(D^{\beta}u)(x)\varphi(x)\,dx.$$

Thus, $\tilde{\Psi}(u) \in W^{k,p}(\Omega)$ and

$$D^{\alpha}\tilde{\Psi}(u)(x) = \sum_{\beta\leq\alpha} p_{\alpha\beta}(x)\tilde{\Psi}(D^{\beta}u)(x).$$

Then

$$\begin{aligned}\int_{\Omega}|D^{\alpha}\tilde{\Psi}(u)(x)|^p\,dx &\leq C\max_{|\beta|\leq|\alpha|}\sup_{x\in\Omega}|p_{\alpha\beta}(x)|^p\int_{\Omega}\underbrace{|D^{\alpha}\tilde{\Psi}(u)(x)|^p}_{=|(D^{\beta}u)(\Psi(x))|^p}\,dx \\ &\leq C\max_{|\beta\leq|\alpha|}\int_{\mathcal{D}}|D^{\beta}u(y)|^p|\det D\Psi^{-1}(y)|\,dy \\ &\leq C\|u\|_{W^{k,p}(\mathcal{D})},\end{aligned}$$

thus $\|\tilde{\Psi}(u)\|_{W^{k,p}(\Omega)} \leq C\|u\|_{W^{k,p}(\mathcal{D})}$. Repeating the argument with Ψ^{-1} in place of Ψ gives the result.