VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 9

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the following statement. Let $Lu \geq f\, (= f)$ in a bounded domain Ω, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and assume that $c \leq 0$. Then, there exists a constant $C > 0$ depending only on the diameter of Ω and on $\frac{\|b\|_{L^\infty(\Omega)}}{\Lambda}$ sub that

$$\sup_{\Omega} u\, (|u|) \leq \sup_{\partial\Omega} u^+\, (|u|) + C \sup_{\Omega} \frac{|f^-|}{\Lambda} \left(\frac{|f|}{\Lambda}\right).$$

($f^- = \inf\{f, 0\}, u^+ = \sup\{u, 0\}$.)

Problem 2. Prove the following statement. Let $Lu = f$ in a bounded domain Ω, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and assume that $c \leq 0$. Let C be the constant of the previous problem and suppose that

$$A = 1 - C \sup_{\Omega} \frac{c^+}{\Lambda} > 0.$$

Then

$$\sup_{\Omega} |u| \leq \frac{1}{A} \left(\sup_{\Omega} |u| + C \sup_{\Omega} \frac{|f|}{\Lambda} \right).$$

2. Solutions

Solution 1. Let Ω lie in the slap $0 < x^1 < d$ and set $L_0 = a^{ij}\partial_i\partial_j + b^i\partial_i$. If $\alpha > \frac{\|b\|_{L^\infty(\Omega)}}{\Lambda} + 1$, then

$$\begin{aligned} L_0 e^{\alpha x^1} &= (\alpha^2 a^{11} + \alpha b^1) e^{\alpha x^1} \\ &\geq (\alpha^2 \Lambda - \alpha \|b\|_{L^\infty(\Omega)}) e^{\alpha x^1} \\ &= (\alpha^2 \Lambda - \alpha\Lambda \frac{\|b\|_{L^\infty(\Omega)}}{\Lambda}) e^{\alpha x^1} \\ &\geq \Lambda. \end{aligned}$$

Set

$$v = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x^1}) \sup_{\Omega} \frac{|f^-|}{\Lambda} \geq 0.$$

Then

$$\begin{aligned} Lv &= -(L_0 e^{\alpha x^1}) \sup_{\Omega} \frac{|f^-|}{\Lambda} + cv \\ &\leq -\sup_{\Omega} |f^-|, \end{aligned}$$

thus

$$L(v - u) \leq -\sup_{\Omega} |f^-| - f \leq 0.$$

We also have $v - u \geq 0$ on $\partial\Omega$. Thus, by one of the corollaries of the maximum principle, $u \leq v$, so

$$\begin{aligned} u &\leq \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x^1}) \sup_{\Omega} \frac{|f^-|}{\Lambda} \\ &\leq \sup_{\partial\Omega} u^+ + (e^{\alpha d} - 1) \sup_{\Omega} \frac{|f^-|}{\Lambda}. \end{aligned}$$

Solution 2. Write $Lu = (L_0 + c)u = f$ as $(L_0 + c^-)u = f - c^+ u =: \tilde{f}$. From the previous problem,

$$\begin{aligned} \sup_{\Omega} |u| &\leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \frac{|\tilde{f}|}{\Lambda} \\ &\leq \sup_{\partial\Omega} |u| + C \left(\sup_{\Omega} \frac{|f|}{\Lambda} + \sup_{\Omega} |u| \sup_{\Omega} \frac{|c^+|}{\Lambda} \right). \end{aligned}$$

Thus

$$\left(1 - C \sup_{\Omega} \frac{|c^+|}{\Lambda} \right) \sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \frac{|f|}{\Lambda}.$$