VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 4

Unless stated otherwise, the notation below is as in class. You can assume that all functions are C^∞ unless stated otherwise.

1. Problems

Problem 1. Prove the differentiation of moving regions formula stated in class:

$$\frac{d}{d\tau}\int_{\Omega(\tau)} f\,dx = \int_{\Omega(\tau)} \partial_\tau f\,dx + \int_{\partial\Omega(\tau)} f v\cdot\nu\,dS. \tag{1.1}$$

(See the class notes for the notation and precise assumptions.) For simplicity, prove (1.1) in the following particular case. Assume that $n = 3$ and that the domains $\Omega(\tau)$ are given by a one-parameter family of one-to-one and onto maps $\varphi = \varphi(\tau, x) : \Omega \to \Omega(\tau) = \varphi(\tau, \Omega)$, where $\Omega := \Omega(0)$ and $\varphi(0, \cdot) = \mathrm{id}_\Omega$, where id_Ω is the identity map on Ω, i.e., $\mathrm{id}_\Omega(x) = x$, $x \in \Omega$.

(a) For each fixed τ, consider the change of variables $x = \varphi(\tau, y)$, so that

$$\int_{\Omega(\tau)} f(\tau, x)\,dx = \int_\Omega f(\tau, \varphi(\tau, y))J(\tau, y)\,dy, \tag{1.2}$$

where $J(\tau, y)$ is the Jacobian of the transformation $x = \varphi(\tau, y)$ for fixed τ.

(b) Show that there exists a on parameter family of vector fields $u(\tau, \cdot)$ such that

$$\partial_\tau\varphi(\tau, x) = u(\tau, \varphi(\tau, x)).$$

(c) Explain why $u = v$ on $\partial\Omega(\tau)$.

(d) Show that

$$\partial_\tau J(\tau, x) = (\mathrm{div} u)(\tau, \varphi(\tau, x))J(\tau, x).$$

(e) Use (1.2) and the above to compute $\frac{d}{d\tau}\int_{\Omega(\tau)} f$, and do an integration by parts to obtain the result.

Problem 2. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^n$. Assume that u_0 and u_1 have their supports in the ball $B_R(0)$ for some $R > 0$. Show that $u = 0$ in the exterior of the region

$$I := \{(t, x) \in (0, \infty) \times \mathbb{R}^n \,|\, x \in B_{R+t}(0)\,\}.$$

I is called a domain of influence for that data on $B_R(0)$ (compare with the $1d$ case).

Problem 3. Let u be a solution to the Cauchy problem for the wave equation and assume that u_0 and u_1 have compact support.

(a) Show that the energy

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^n} \left[(\partial_t u)^2 + |\nabla u|^2\right]\,dx$$

is well-defined.

(b) Show that

$$E(t) = E(0),$$

i.e., the energy is conserved.

Problem 4. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^3$ with compactly supported data (i.e., u_0 and u_1 have compact support).

(a) Show that there exists a constant $C > 0$, depending on u_0 and u_1, such that

$$|u(t,x)| \leq \frac{C}{t}, \tag{1.3}$$

for $t \geq 1$ and $x \in \mathbb{R}^3$. Thus, for each fixed x, u approaches zero as $t \to \infty$, i.e., solutions decay in time.

Hint: Use the formula for solutions in $n = 3$. Since the data has compact support, it vanishes outside $B_R(0)$ for some $R > 0$. This implies an estimate for the area of the largest region within $B_t(x)$ where the data is non-trivial.

(b) Is the estimate (1.3) sharp? (I.e., can it be improved to show that solutions decay faster in time than $\frac{1}{t}$?)

(c) Do we still get decay if the data does not have compact support?

Problem 5. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $1d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(t-s,y)\,dyds. \tag{1.4}$$

To do so, first use D'Alembert's formula to conclude that

$$u_s(t,x) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy.$$

Use the definition of u in terms of u_s and change variables to conclude (1.4).

Problem 6. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $3d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\,dy. \tag{1.5}$$

(The integrand in (1.5) is known as the retarded potential.) To do so, first use Kirchhoff's formula for solutions in $n = 3$ to conclude that

$$u_s(t,x) = \frac{t-s}{\operatorname{vol}(\partial B_{t-s}(x))}\int_{\partial B_{t-s}(x)} f(s,y)\,dS(y).$$

Use the definition of u in terms of u_s and change variables to conclude (1.5).

Problem 7. Show that there exists a constant $C > 0$ such that for any solution u to the $3d$ wave equation it holds that

$$|u(t,x)| \leq \frac{C}{t}\int_{\mathbb{R}^3} (|D^2u_0(y)| + |Du_0(y)| + |u_0(y)| + |Du_1(y)| + |u_1(y)|)\,dy$$

for $t \geq 1$.

Hint: Use Kirchhoff's formula, note that for any function f we have

$$f(y) = f(y)\frac{y-x}{t} \cdot \frac{y-x}{t}$$

on $\partial B_t(x)$, and use one of Green's identities.

Problem 8. Consider continuous dependence on the data for the wave equation in $3d$, where smallness on the data part is measured with respect to the norm

$$\|f\|_2 := \int_{\mathbb{R}^3} (|D^2 f(y)| + |Df(y)| + |f(y)|)\, dy.$$

Give a precise formulation of the continuous dependence on the data and prove your statement, i.e., a statement saying that two solutions are close if their corresponding initial data are close.

Hint: Use the estimate of problem 7 as a basis for your statement, and give a similar proof (now you have to also account for $t < 1$).

2. Solutions

Solution 1. (a) This is simply the change of variables formula from calculus.

(b) For each fixed x, the map $\tau \mapsto \varphi(\tau, x)$ is a curve in $\mathbb{R}^3$. $\partial_\tau \varphi(\tau, x)$ is, therefore, the tangent vector to this curve at $\varphi(\tau, x)$ at time τ. The collection of all such tangent vectors, as τ and x vary, forms the vector field u.

(c) The map φ sends $\partial\Omega$ onto $\partial\Omega(\tau)$ for each τ. Since $\partial_\tau \varphi(\tau, x)$ is the velocity at time τ of the particle that started at $x \in \Omega$ at time zero, $u(\tau, \varphi(\tau, x))$ is the velocity of $\partial\Omega(\tau)$ at the point $\varphi(\tau, x) \in \partial\Omega(\tau)$.

(d) According to the notation of part (a), we set

$$\varphi^i_j = \frac{\partial}{\partial y^j}\varphi^i, \; \partial_j u^i = \frac{\partial}{\partial x^j} u^i,$$

where we considered $\varphi = (\varphi^1, \varphi^2, \varphi^3)$. In particular, note that when we write $\varphi^i_j = \partial_j \varphi^i$ the derivative is always with respect to $y \in \Omega$, whereas when we write $\partial_j u^i$ the derivative is always with respect to $x \in \Omega(\tau)$.

Recall the following formula for the determinant of a $n \times n$ matrix a with entries $a^i_j = a^{\text{row}}_{\text{column}}$:

$$\det(a) = \frac{1}{n!} \epsilon_{i_1 \cdots i_n} \epsilon^{j_1 \cdots j_n} a^{i_1}_{j_1} \cdots a^{i_n}_{j_n}.$$

In our case, this gives

$$J(\tau, y) = \frac{1}{3!} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \varphi^{i_1}_{j_1} \varphi^{i_2}_{j_2} \varphi^{i_3}_{j_3}.$$

Recall that the definition of J involves an absolute value, which we can omit here since $J > 0$ because $J(0, \cdot) > 0$. Compute

$$\begin{aligned} \partial_\tau \varphi^i_j &= \partial_j \partial_\tau \varphi^i \\ &= \frac{\partial}{\partial y^j} u^i \\ &= \partial_\ell u^i \varphi^\ell_j, \end{aligned}$$

where in the second equality we used (b) and in the third one the chain rule. Therefore

$$\partial_\tau J(\tau, y) = \frac{1}{3!} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_\ell u^{i_1} \varphi^\ell_{j_1} \varphi^{i_2}_{j_2} \varphi^{i_3}_{j_3} + \varphi^{i_1}_{j_1} \partial_\ell u^{i_2} \varphi^\ell_{j_2} \varphi^{i_3}_{j_3} + \varphi^{i_1}_{j_1} \varphi^{i_2}_{j_2} \partial_\ell u^{i_3} \varphi^\ell_{j_3}). \tag{2.1}$$

Because $\epsilon_{i_1 i_2 i_3}$ is non-zero only for $i_1 i_2 i_3$ all different from each other, for each triple $i_1 i_2 i_3$, the term $\epsilon_{i_1 i_2 i_3} \partial_\ell u^{i_1} \varphi^\ell_{j_1} \varphi^{i_2}_{j_2} \varphi^{i_3}_{j_3}$ is non-zero only when $\ell = i_1$. Similarly for the second and third terms on the RHS of (2.1), and we obtain

$$\partial_\tau J(\tau, y) = \frac{1}{3!} \sum_{\substack{i_1,i_2,i_3=1 \\ j_1,j_2,j_3=1}}^{3} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_{i_1} u^{i_1} + \partial_{i_2} u^{i_2} + \partial_{i_3} u^{i_3}) \varphi^{i_1}_{j_1} \varphi^{i_2}_{j_2} \varphi^{i_3}_{j_3}.$$

Because the summand is non-zero only if $i_1 i_2 i_3$ are all different from each other, the term in parenthesis is always equal to $\partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3 = \mathrm{div} u$, which gives the result.

(e) We have

$$\begin{aligned}
\frac{d}{d\tau} \int_{\Omega(\tau)} f\, dx &= \partial_\tau \int_\Omega f(\tau, \varphi(\tau, y)) J(\tau, y)\, dy \\
&= \int_\Omega \left(\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot \partial_\tau \varphi(\tau, y) J(\tau, y) + f(\tau, \varphi(\tau, y)) \partial_\tau J(\tau, y)\right) dy \\
&= \int_\Omega \big(\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot u(\tau, \varphi(\tau, y)) J(\tau, y) \\
&\qquad + f(\tau, \varphi(\tau, y)) (\mathrm{div} u)(\tau, \varphi(\tau, y)) J(\tau, y)\big)\, dy \\
&= \int_{\Omega(\tau)} \left(\partial_\tau f(\tau, x) + \nabla f(\tau, x) \cdot u(\tau, x) + f(\tau, x)(\mathrm{div} u)(\tau, x)\right) dx \\
&= \int_{\Omega(\tau)} \left(\partial_\tau f(\tau, x) - f(\tau, x)(\mathrm{div} u)(\tau, x) + f(\tau, x)(\mathrm{div} u)(\tau, x)\right) dx \\
&\quad + \int_{\partial\Omega(\tau)} f(\tau, x) u(\tau, x) \cdot \nu(\tau, x)\, dS(x) \\
&= \int_{\Omega(\tau)} \partial_\tau f(\tau, x)\, dx + \int_{\partial\Omega(\tau)} f(\tau, x) v(\tau, x) \cdot \nu(\tau, x)\, dS(x).
\end{aligned}$$

Above, we the steps are as follows: in the second line we used the product rule and the chain rule; in the third line we used (b) and (d); on the fourth line, we changed variables back to x; on the fifth line we integrated ∇f by parts (equivalently, used on of the Green identities); on the last line, we used (c).

Solution 2. Let $(t, x) \notin I$. Then $K^-_{t,x} \cap I = \varnothing$, and the result follows from the finite-propagation speed for the wave equation.

Solution 3. (a) By question 2, the solution u has compact support for each fixed t.

(b) For each t_0 and $\varepsilon > 0$, there exists, by (a), a $R_* > 0$ such that $u(t, x) = 0$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $|x| \geq R_*$. We now follow the proof of the finite-propagation speed property for the wave equation (see the class notes) using the ball B_{R_*}, and observe the following. In that proof, we did an integration by parts, and controlled the boundary term using the Cauchy-Schwarz inequality. Here, this boundary term vanishes identically by the foregoing. We obtain therefore a sequence of equalities (rather than inequalities as in the proof done in class), which then gives the result.

Solution 4. (a) The solution is given by

$$u(t, x) = \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y - x))\, dS(y).$$

Since the data is compactly supported, there exists a $R > 0$ such that $u_0(x) = 0$ and $u_1(x) = 0$ for $|x| \geq R$, so that

$$u(t, x) = \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y - x))\, dS(y).$$

Because the data is compactly supported, we have $|u_0|, |u_1|, |\nabla u_0| \le C$ for some $C > 0$, so that

$$
\begin{aligned}
|u(t,x)| &\le \frac{C}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + |y-x|)\, dS \\
&= \frac{C}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + \frac{t|y-x|}{t})\, dS \\
&\le \frac{C}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + t)\, dS \\
&\le \frac{C(1+t)}{t^2} \int_{\partial B_t(x) \cap B_R(0)} dS,
\end{aligned}
$$

where we used that $|y-x|/t = 1$ since $y \in B_t(x)$ and that $\operatorname{vol}(\partial B_t(x)) = 4\pi t^2$. Because $\partial B_t(x) \cap B_R(0)$ has area at most $4\pi R^2$, we have the result.

(b) Yes, it cannot be improved for arbitrary solutions of the wave equation. To see this, take $u_0 = 0$ and u_1 to be a non-negative compactly supported function that is equal to 1 on $B_1(0)$. Then

$$
\begin{aligned}
u(t,x) &= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} t u_1(y)\, dS(y) \\
&= \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y)\, dS(y) + \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \setminus (B_t(x) \cap B_1(0))} u_1(y)\, dS(y).
\end{aligned}
$$

Note that the second term on the RHS is always non-negative, thus

$$
u(t,x) \ge \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y)\, dS(y) = \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} dS.
$$

For any x on the boundary of the lightcone, i.e., $|x| = t$, and such that $|x| \ge 1$, we have that the area of $\partial B_t(x) \cap B_1(0)$ is $\ge C > 0$, so that $u(t,x) \ge C/t$.

(c) Not necessarily, e.g., take $u_0 = 0$ and $u_1 = 1$, then $u(t,x) = t$ is the solution.

Solution 5. Using D'Alembert's formula, we find

$$
u_s(t,x) = \frac{1}{2} \int_{x-t+s}^{x+t-s} f(s,y)\, dy,
$$

where we used the fact that D'Alembert's formula was derived for data at $t = 0$; for data at $t = s$ we have to replace t by $t - s$ in the limits of integration. Thus

$$
u(t,x) = \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\, dyds = \frac{1}{2} \int_0^t \int_{x-z}^{x+z} f(t-z,y)\, dydz,
$$

where we made the change $s = t - z$.

Solution 6. Kirchhoff's formula gives

$$
u_s(t,x) = \frac{1}{\operatorname{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} (t-s) f(s,y)\, dS(y).
$$

Thus

$$\begin{aligned}
u(t,x) &= \int_0^t \frac{t-s}{\operatorname{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} f(s,y)\, dS(y) ds \\
&= \frac{1}{4\pi} \int_0^t \int_{\partial B_{t-s}(x)} \frac{f(s,y)}{t-s}\, dS(y) ds \\
&= \frac{1}{4\pi} \int_0^t \int_{\partial B_r(x)} \frac{f(t-r,y)}{r}\, dS(y) dr \\
&= \frac{1}{4\pi} \int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\, dy,
\end{aligned}$$

where we made the change of variables $r = t - s$ and then wrote $r = |y-x|$.

Solution 7. We have

$$u(t,x) = \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \left(u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y-x)\right)\, dS(y).$$

The unit outer normal to $\partial B_t(x)$ is $\nu = (y-x)/t$, so that $\nu \cdot \nu = \frac{y-x}{t} \cdot \frac{y-x}{t} = 1$. Therefore, using this and Green's identities,

$$\begin{aligned}
\frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) &= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y) \nu \cdot \frac{y-x}{t}\, dS(y) \\
&= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{B_t(x)} \operatorname{div}_y \left(u_0(y) \frac{y-x}{t} \right) dy \\
&= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{B_t(x)} \left(\nabla u_0(y) \cdot \frac{y-x}{t} + u_0(y) \frac{3}{t} \right) dy,
\end{aligned}$$

so that

$$\begin{aligned}
\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) \right| &\le \frac{C}{t^2} \int_{B_t(x)} \left(|\nabla u_0(y)| + |u_0(y)|\right)\, dy \\
&\le \frac{C}{t^2} \int_{\mathbb{R}^3} \left(|\nabla u_0(y)| + |u_0(y)|\right)\, dy.
\end{aligned}$$

A similar inequality holds for the u_1 integral (with an extra factor of t), and for ∇u_0:

$$\begin{aligned}
\frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot (y-x)\, dS(y) &= \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot \nu\, dS(y) \\
&= \frac{1}{4\pi t} \int_{B_t(x)} \Delta u_0(y)\, dy,
\end{aligned}$$

so that

$$\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot (y-x)\, dS(y) \right| \le \frac{C}{t} \int_{\mathbb{R}^3} |D^2 u_0(y)|\, dy.$$

Combining the foregoing produces the result.

Solution 8. We formulate it as follows. Let (u_0, u_1) and (v_0, v_1) be two data sets for the wave equation, and let u and v be the respective solutions. Solutions depend continuously on the data if given $\varepsilon > 0$ and $t > 0$, there exists a $\delta > 0$ such that if

$$\|u_0 - v_0\|_2 + \|u_1 - v_1\|_2 < \delta,$$

then

$$|u(t,x) - v(t,x)| < \varepsilon$$

for all $x \in \mathbb{R}^3$.

We now prove the statement. Set $w_0 = u_0 - v_0$, $w_1 = u_1 - v_1$, and $w = u - v$. By Kirchhoff's formula:

$$w(t,x) = \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (w_0(y) + t w_1(y) + \nabla w_0(y) \cdot (y - x)) \, dS(y).$$

Proceeding as in problem 7, we find

$$\left| \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} w_0(y) \, dS(y) \right| \leq \frac{C}{t^2} \int_{\mathbb{R}^3} (|\nabla w_0(y)| + |w_0(y)|) \, dy,$$

$$\left| \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} w_1(y) \, dS(y) \right| \leq \frac{C}{t} \int_{\mathbb{R}^3} (|\nabla w_1(y)| + |w_1(y)|) \, dy,$$

and

$$\left| \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla w_0(y) \cdot (y - x) \, dS(y) \right| \leq \frac{C}{t} \int_{\mathbb{R}^3} |D^2 w_0(y)| \, dy.$$

Combining the above we find

$$|w(t,x)| \leq C \max\{\frac{1}{t}, \frac{1}{t^2}\}(\|w_0\|_2 + \|w_1\|_2),$$

which implies the result.