VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 11

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. In class, we defined the concept of a function with local compact support in x, and discussed that a smooth function $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^d$ can be regarded as an element of $C^m\left(\mathbb{R}, H^k(\mathbb{R}^n, \mathbb{R}^d)\right)$ for any $m, k \geq 0$. Show that this is not the case if f is assumed only to be such that for each fixed t, $f(t, \cdot)$ has compact support.

Hint: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and define f by

$$f(t,x) = \begin{cases} \varphi(x^1 - \frac{1}{t}, x^2, \dots, x^n), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Problem 2. Verify the inequalities $\mathcal{M}_k[v_0] \leq \mathcal{C}$ and $\mathcal{M}_k[v_1] \leq \mathcal{C}$ in the proof of local existence and uniqueness of solutions to quasilinear wave equations.

Problem 3. Let $\{f_i\} \subset H^k(\mathbb{R}^n)$ be a bounded sequence that converges to f in $H^\ell(\mathbb{R}^n)$, $\ell < k$. Show that $f \in H^k(\mathbb{R}^n)$.

2. Solutions

Solution 1. Observe that f is smooth for $t > 0$ and for $t < 0$. For each $(0, x)$, there exists a neighborhood U of $(0, x)$ in $\mathbb{R} \times \mathbb{R}^n$ such that $f = 0$ in U. Thus, f is smooth. For fixed t, $f(t, \cdot)$ has compact support. For $t \leq 0$, $\|f(t, \cdot)\|_{L^2(\mathbb{R}^n)} = 0$. But for $t > 0$, $\|f(t, \cdot)\|_{L^2(\mathbb{R}^n)} > 0$. Thus $f \notin C^0\left(\mathbb{R}, H^0(\mathbb{R}^n, \mathbb{R})\right)$.

Solution 2. We have $\mathcal{M}_k[v_0] = \mathcal{M}_k[u_{0,0}] \leq C_0 + 1$ by assumption, so we can choose $\mathcal{C} \geq C_0 + 1$. For v_1, we need $\mathcal{N}[v_{i-1}] = \mathcal{N}[v_0] \leq z_I(\mathcal{C})$. In the proof, this was obtained using the induction hypothesis for v_{i-2}, which would give v_{-1} here, which has not been defined. But we have $\mathcal{N}[v_0] \leq z_I(\mathcal{C})$ directly from the fact that v_0 is constant in time and from Sobolev embedding.

Solution 3. Since the sequence is bounded in $H^k(\mathbb{R}^n)$ it converges weakly to a limit $\tilde{f} \in H^k(\mathbb{R}^n)$. Because $H^k(\mathbb{R}^n) \hookrightarrow H^\ell(\mathbb{R}^n)$ compactly, f_i converges to $\tilde{f}$ in $H^\ell(\mathbb{R}^n)$. Uniqueness of the limit gives $\tilde{f} = f$.