VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 8

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the uniqueness statement in the proof of the "Riesz representation for Sobolev spaces" (the part that was not done in class).

Problem 2. Prove that

$$\|D^{\alpha_1}u_i\cdots D^{\alpha_\ell}u_\ell\|_{L^2(\mathbb{R}^n)} \le C\sum_{i=1}^{\ell}\|D^k u_i\|_{L^2(\mathbb{R}^n)}\prod_{j\neq i}\|u_j\|_{L^\infty(\mathbb{R}^n)},$$

for $u_i \in H^k(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and $\sum_i |\alpha_i| = k$.
Hint: You can use, without proof, the Gagliardo-Nirenberg inequality

$$\|D^j u\|_{L^{\frac{2r}{j}}(\mathbb{R}^n)} \le C\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{j}{r}}\|D^r u\|_{L^2(\mathbb{R}^n)}^{\frac{j}{r}}.$$

Problem 3. In the context of Egorov's example, prove the lemma that reduces the necessary condition for existence of weak solutions to

$$\|v\|_0 \le C\|L^* v\|_N,$$

$v \in C_c^\infty(\Omega)$.

2. Solutions

Solution 1. We follow the notation used in class. Suppose the conclusion holds for v_1 and v_2 attaining the minimum, so

$$\|v_1\|_{L^{p'}(\Omega_{(k)})} = \|f\|_{(W^{k,p}(\Omega))'} = \|v_2\|_{L^{p'}(\Omega_{(k)})} = 1,$$

where we can assume $= 1$ upon redefining f as $\frac{f}{\|f\|_{(W^{k,p}(\Omega))'}}$, and for all $u \in W^{k,p}(\Omega)$,

$$f(u) = \sum_{|\alpha|\le k}\langle v_1, D^\alpha u\rangle = \sum_{|\alpha|\le k}\langle v_2, D^\alpha u\rangle.$$

First, we claim that there exists a unique $x \in X$ such that

$$f^*(x) = \|x\|_{L^p(\Omega_{(k)})} = 1.$$

Since $\|f\|_{(W^{k,p}(\Omega))'} = \|f^*\|_{X'} = 1$, there exists $\{x_i\} \subset X$ such that $\|x_i\|_{L^p(\Omega_{(k)})} = 1$ and $|f^*(x_i)| \to 1$; we can further assume that $f^*(x_i) \to 1$ by modifying the sequence if necessary. Because $L^p(\Omega_{(k)})$ is uniformly convex for $1 < p < \infty$, given $0 < \varepsilon \le 2$, there exists a $\delta > 0$ such that if $\|x_i - x_j\|_{L^p(\Omega_{(k)})} \ge \varepsilon$ then $\|\frac{x_i+x_j}{2}\|_{L^p(\Omega_{(k)})} \le 1-\delta$, thus if $\|\frac{x_i+x_j}{2}\|_{L^p(\Omega_{(k)})} > 1-\delta$ we must have $\|x_i - x_j\|_{L^p(\Omega_{(k)})} < \varepsilon$. For large i we have $f^*(x_i) > 1-\delta$ thus for large i,j we also have $f^*(\frac{x_i+x_j}{2}) > 1-\delta$. Hence, as f^* is continuous with norm 1, $1-\delta < f^*(\frac{x_i+x_j}{2}) \le \|\frac{x_i+x_j}{2}\|_{L^p(\Omega_{(k)})}$. Therefore, $\|x_i - x_j\|_{L^p(\Omega_{(k)})} < \varepsilon$ and $\{x_i\}$ is Cauchy, thus $x_i \to x$ in $L^p(\Omega_{(k)})$ and $x \in X$ since X is closed. Clearly $\|x\|_{L^p(\Omega_{(k)})} = 1$

and $f^*(x) = 1$. To obtain uniqueness, if there are two such x's, say, x_1 and x_2, we can apply the above argument to the sequence $\{x_1, x_2, x_1, x_2, \dots\}$, which must converge.

Since v_1 and v_2 are two representatives of f^*, we have

$$f^*(x) = 1 = \sum_{|\alpha|\le k} \langle (v_1)_\alpha, x_\alpha\rangle = \sum_{|\alpha|\le k} \langle (v_2)_\alpha, x_\alpha\rangle.$$

Consider the following claim: given $w \in L^p(\Omega_{(k)})$ with $\|w\|_{L^p(\Omega_{(k)})} = 1$, there exists at most one $\ell \in (L^p(\Omega_{(k)}))'$ such that $\|\ell\|_{(L^p(\Omega_{(k)}))'} = 1$ and $\ell(w) = 1$.

Let $\tilde{v}_1$ and $\tilde{v}_2$ be the extensions of v_1 and v_2, considered as linear functionals on X, to $L^p(\Omega_{(k)})$ given by Hahn-Banach. Thus $\|\tilde{v}_1\|_{(L^p(\Omega_{(k)}))'} = 1 = \|\tilde{v}_2\|_{(L^p(\Omega_{(k)}))'}$ (observe taht even though $\tilde{v}_1 = f^* = \tilde{v}_2$ on X, we cannot claim from this that $\tilde{v}_1 = \tilde{v}_2$ because the Hanh-Banach extensions might not be unique), and by the foregoing we have $\tilde{v}_1(x) = 1 = \tilde{v}_2(x)$. Thus $\tilde{v}_1 = \tilde{v}_2$ by the above claim.

It remains to prove the above claim. Suppose that there are two such $\ell's$, ℓ_1 and ℓ_2, $\ell_1 \ne \ell_2$. Thus $\ell_1(u) \ne \ell_2(u)$ for some $u \in L^p(\Omega_{(k)})$. We can assume that $\ell_1(u) - \ell_2(u) = 2$ upon replacing u by a suitable multiple of itself, and that $\ell_1(u) = 1$ and $\ell_2(u) = -1$ upon replacing u with its sum with a suitable multiple of w. Thus

$$\begin{aligned} \ell_1(w + tu) &= 1 + t, \\ \ell_2(w - tu) &= 1 + t, \end{aligned}$$

$t > 0$. Since $\|\ell_1\|_{(L^p(\Omega_{(k)}))'} = 1 = \|\ell_2\|_{(L^p(\Omega_{(k)}))'}$,

$$\begin{aligned} 1 + t = \ell_1(w + tu) &\le \|w + tu\|_{L^p(\Omega_{(k)})}, \\ 1 + t = \ell_1(w - tu) &\le \|w + tu\|_{L^p(\Omega_{(k)})}. \end{aligned}$$

Recall the L^p-parallelogram inequalities:

$$\begin{aligned} &\|\frac{a+b}{2}\|^p_{L^p} + \|\frac{a-b}{2}\|^p_{L^p} \ge \frac{1}{2}\|a\|^p_{L^p} + \frac{1}{2}\|b\|^p_{L^p},\ 1 < p \le 2, \\ &\|\frac{a+b}{2}\|^{p'}_{L^p} + \|\frac{a-b}{2}\|^{p'}_{L^p} \ge (\frac{1}{2}\|a\|^p_{L^p} + \frac{1}{2}\|b\|^p_{L^p})^{p'-1},\ 2 \le p < \infty. \end{aligned}$$

If $1 < p \le 2$, we get

$$\begin{aligned} 1 + t^p\|u\|^p_{L^p(\Omega_{(k)})} &= \|\frac{(w+tu)+(w-tu)}{2}\|^p_{L^p(\Omega_{(k)})} + \|\frac{(w+tu)-(w-tu)}{2}\|^p_{L^p(\Omega_{(k)})} \\ &\ge \frac{1}{2}\|w+tu\|^p_{L^p(\Omega_{(k)})} + \frac{1}{2}\|w-tu\|^p_{L^p(\Omega_{(k)})} \\ &\ge (1+t)^p, \end{aligned}$$

which cannot be true for all $t > 0$. If $2 \le p \le \infty$, we apply the second inequality to get

$$\begin{aligned} 1 + t^{p'}\|u\|^{p'}_{L^p(\Omega_{(k)})} &= \|\frac{(w+tu)+(w-tu)}{2}\|^{p'}_{L^p(\Omega_{(k)})} + \|\frac{(w+tu)-(w-tu)}{2}\|^{p'}_{L^p(\Omega_{(k)})} \\ &\ge (\frac{1}{2}\|w+tu\|^p_{L^p(\Omega_{(k)})} + \frac{1}{2}\|w-tu\|^p_{L^p(\Omega_{(k)})})^{p'-1} \\ &\ge (1+t)^{p'}, \end{aligned}$$

which again is an impossibility.

Solution 2. From Hölder's inequality and the product rule,

$$\begin{aligned} \|D^\alpha(uv)\|_{L^2(\mathbb{R}^n)} &\le \sum_{\beta\le\alpha} C\|D^\beta u D^{\alpha-\beta} v\|_{L^2(\mathbb{R}^n)} \\ &\le C\sum_{\beta\le\alpha} \|D^\beta u\|_{L^{\frac{2k}{|\beta|}}(\mathbb{R}^n)} \|D^{\alpha-\beta} v\|_{L^{\frac{2k}{|\alpha-\beta|}}(\mathbb{R}^n)} \end{aligned}$$

The Gagliardo-Niremberg inequality gives

$$\begin{aligned}\|D^\alpha(uv)\|_{L^2(\mathbb{R}^n)} &\leq C\sum_{\beta\leq\alpha}\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\beta|}{k}}\|D^k u\|_{L^2(\mathbb{R}^n)}^{\frac{|\beta|}{k}}\|v\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\alpha-\beta|}{k}}\|D^k v\|_{L^2(\mathbb{R}^n)}^{\frac{|\alpha-\beta|}{k}}\\ &\leq C\sum_{\beta\leq\alpha}\left(\|u\|_{L^\infty(\mathbb{R}^n)}\|D^k v\|_{L^2(\mathbb{R}^n)}\right)^{\frac{|\alpha-\beta|}{k}}\left(\|D^k u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^\infty(\mathbb{R}^n)}\right)^{\frac{|\beta|}{k}}\\ &\leq C(\|u\|_{L^\infty(\mathbb{R}^n)}\|D^k v\|_{L^2(\mathbb{R}^n)}+\|v\|_{L^\infty(\mathbb{R}^n)}\|D^k u\|_{L^2(\mathbb{R}^n)})\end{aligned}$$

which implies the result.

Solution 3. Using that we now established $H{-}k(\Omega) \approx (H^k(\Omega))'$ for $k \in \mathbb{Z}$ (this had been established initially for $k \geq 0$), the necessary condition for existence be be extended for $s, t \in \mathbb{Z}$. Thus, there exist $s, t \in \mathbb{Z}$ such that

$$\|v\|_s \leq C\|L^* v\|_t.$$

If $s \geq 0$, then we can choose $N \geq t$. Otherwise, we can assume $t \geq s$ since if $t < s$ then we can choose $\tilde{t} \geq s$ and work with $\tilde{t}$ (since $\|L^*v\|_t \leq \|L^*v\|_{\tilde{t}}$ then). Because $D_x^\alpha v \in C_c^\infty(\Omega)$ if $v \in C_c^\infty(\Omega)$, we can apply the inequality to $D_x^\alpha v$ to get

$$\|D_x^\alpha v\|_s \leq C\|L^* D_x^\alpha v\|_t \leq C\|D_x^\alpha L^* v\|_t \leq C\|D_x^\alpha L^* v\|_{t+|\alpha|},$$

where we used that $L^*v = \partial_t^2 v - a(t)\partial_x^2 v - b(t)\partial_x v$. We also have

$$\|\partial_t^2 v\|_{s-1} \leq C(\|L^*v\|_{s-1} + \|\partial_x^2 v\|_{s-1} + \|\partial_x v\|_{s-1}) \leq \|L^*v\|_{t+1},$$

where we used $\|L^*v\|_{s-1} \leq \|L^*v\|_{t+1}$ by $s \leq t$ and $\|\partial_x^2 v\|_{s-1} + \|\partial_x v\|_{s-1} \leq \|L^*v\|_{t+1}$ by the above. Then

$$\begin{aligned}\|v\|_{s+1} \leq& C(\|v\|_s + \|\partial_t^2 v\|_{s-1} + \|\partial_x^2 v\|_{s-1})\\ &\leq C(\|L^*v\|_t + \|L^*v\|_{t+1})\\ &\leq \|L^*v\|_{t+1}.\end{aligned}$$

Iterating this argument gives the result.