VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 10

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the following statement used in the proof of existence of solutions to linear first-order symmetric hyperbolic systems. Let $u \in L^\infty\left(J, H^k(\mathbb{R}^n)\right)$, where k is a non-negative integer and J is an open interval. Then there exists a $\tilde{u} \in L^2_{loc}(J \times \mathbb{R}^n)$ that is k times weakly differentiable with respect to x and with derivatives in $L^2_{loc}(J \times \mathbb{R}^n)$ and such that

$$\langle \varphi, u \rangle = \int_{J\times\mathbb{R}^n} \varphi \tilde{u} \, dx \, dt,$$

for all $\varphi \in C_c^\infty(J \times \mathbb{R}^n)$.

2. Solutions

Solution 1. Consider first $k = 0$. Let u_i be a sequence of step functions converging point-wise a.e. to u. Thus

$$u_i = \sum_{\ell=1}^{N_i} f_{i,\ell} \chi_{A_{i,\ell}},$$

where $A_{i,\ell} \subset J$ are measurable sets and $f_{i,\ell} \in L^2(\mathbb{R}^n)$. u_ℓ defines a $dt \times dx$-measurable function which is also measurable in $J \times \mathbb{R}^n$ (i.e., measurable with respect to the $(n+1)$-dimensional Lebesgue measure, since the $(n+1)$-dimensional Lebesgue measure is the completion of the $1 \times n$ measure).

Let $B_i := \{t \in J \mid \|u_i(t)\|_{L^2(\mathbb{R}^n)} \leq 2\|u(t)\|_{L^2(\mathbb{R}^n)}\}$. Set $u_i' = \chi_{B_i} u_i$. Then, u_i' converges dt-a.e. to u with respect to L^2 and is measurable in $J \times \mathbb{R}^n$. Given a compact set $K \subset J \times \mathbb{R}^n$, let K_1 be its projection onto J and set $U_i = u_i' \chi_K$; U_i is $dt \times dx$-measurable. We have

$$\left(\int_{\mathbb{R}^n} |U_i(t,x) - U_j(t,x)| \, dx \right)^{\frac{1}{2}} \leq \chi_{K_1}(t) \|u_i'(t) - u_j'(t)\|_{L^2(\mathbb{R}^n)}.$$

Consider

$$\|u_i'(t) - u_j'(t)\|_{L^2(\mathbb{R}^n)} \leq \|u_i'(t) - u(t)\|_{L^2(\mathbb{R}^n)} + \|u(t) - u_j'(t)\|_{L^2(\mathbb{R}^n)}.$$

Each term on the RHS converges to zero point-wise a.e. (in t) and is bounded by a function in $L^\infty(J)$; thus, dominated convergence implies that $\{U_i\}$ is a Cauchy sequence in $L^2(K)$. Thus, for each K we have an element $U_K \in L^2(J \times \mathbb{R}^n)$. Taking an increasing sequence of compact sets we obtain a locally square-integrable function U in $J \times \mathbb{R}^n$. We finally observe that

$$\langle \varphi, u \rangle = \lim_{i\to\infty} \langle \varphi, u_i' \rangle = \int_{J\times\mathbb{R}^n} \varphi U \, dt \, dx$$

for all $\varphi \in C_c^\infty(J \times \mathbb{R}^n)$. This gives the result for $k = 0$.

For $k \geq 1$, we apply the above to the function $D^{\vec{\alpha}} u \in L^\infty\left(J, L^2(\mathbb{R}^n)\right)$ to obtain U_α such that

$$\langle \varphi, D^{\vec{\alpha}} u \rangle = \int_{J\times\mathbb{R}^n} \varphi U_\alpha \, dt \, dx$$

for all $\varphi \in C_c^\infty(J \times \mathbb{R}^n)$, which gives the result.