Stony Brook University.

MAT 127 — Calculus C, Spring 12.

Examples for section 8.5

Question: Find the radius of convergence and interval of convergence of the series.

$$(a)\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}} \quad (b)\sum_{n=1}^{\infty}(-1)^n\frac{n^2x^n}{2^n} \quad (c)\sum_{n=1}^{\infty}\frac{(4x+1)^n}{n^2}$$

Solutions.

(a) Identify $c_n = \frac{1}{\sqrt{n}}$, $a = 0$ and $a_n = \frac{x^n}{\sqrt{n}}$. Use the ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{\sqrt{n+1}}}{\frac{x^n}{\sqrt{n}}}\right| = \left|\frac{x^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{x^n}\right| = \frac{\sqrt{n}}{\sqrt{n+1}}|x|$$

So

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}|x| = |x|$$

because $\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1$. Now set $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ to find

$$|x| < 1$$

Therefore the radius of convergence is $R = 1$. Since $a = 0$, the interval of radius 1 centered at $a = 0$ is $(-1, 1)$. To find the interval of convergence we need to plug at the endpoints. Plug $x = -1$ to find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

This is an alternating series with $b_n = \frac{1}{\sqrt{n}}$. By the alternating series test, it converges.

Plugging $x = 1$ we find

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

This is a p-series with $p = \frac{1}{2}$. By the p-series test, it diverges. So the interval of convergence is $[-1, 1)$.

(b) Identify $c_n = \frac{(-1)^n n^2}{2^n}$, $a = 0$, $a_n = \frac{(-1)^n n^2}{2^n}x^n$. Use ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^2x^{n+1}}{2^{n+1}}}{\frac{(-1)^n n^2 x^n}{2^n}}\right| = \left|\frac{(-1)^{n+1}(n+1)^2x^{n+1}}{2^{n+1}}\frac{2^n}{(-1)^n n^2 x^n}\right| = \frac{1}{2}\frac{(n+1)^2}{n^2}|x|$$

So

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\frac{(n+1)^2}{n^2}|x| = \frac{1}{2}|x|$$

because $\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = 1$. Now set $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ to find

$$|x| < 2$$

Therefore the radius of convergence is $R = 2$. Since $a = 0$, the interval of radius 2 centered at $a = 0$ is $(-2, 2)$. To find the interval of convergence we need to plug at the endpoints. Plug $x = -2$ to find

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2(-2)^n}{2^n} = \sum_{n=1}^{\infty} n^2$$

which diverges by the divergence test. Analogously plugging $x = 2$ we find

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2(2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n^2$$

which also diverges by the divergence test. Hence the interval of convergence is $(-2, 2)$.

(c) First notice that we have to write the series as

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{[4(x+\frac{1}{4})]^n}{n^2} = \sum_{n=1}^{\infty} \frac{4^n(x+\frac{1}{4})^n}{n^2}$$

since the term with power of x has to be of the form $(x-a)^n$, i.e., without a number multiplying the x. Now identify $c_n = \frac{4^n}{n^2}$, $a = -\frac{1}{4}$ (a is negative because $(x-a)^n = (x+\frac{1}{4})^n$), and $a_n = \frac{4^n(x+\frac{1}{4})^n}{n^2}$. The ratio test gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{4^{n+1}(x+\frac{1}{4})^{n+1}}{(n+1)^2}}{\frac{4^n(x+\frac{1}{4})^n}{n^2}}\right| = \left|\frac{4^{n+1}(x+\frac{1}{4})^{n+1}}{(n+1)^2} \frac{n^2}{4^n(x+\frac{1}{4})^n}\right| = 4\frac{n^2}{(n+1)^2}|x+\frac{1}{4}|$$

So

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 4\frac{n^2}{(n+1)^2}|x+\frac{1}{4}| = 4|x+\frac{1}{4}|$$

because $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$. Now set $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ to find

$$|x+\frac{1}{4}| < \frac{1}{4}$$

Therefore the radius of convergence is $R = \frac{1}{4}$. Since $a = -\frac{1}{4}$, the interval of radius $\frac{1}{4}$ centered at $a = -\frac{1}{4}$ is $(-\frac{1}{2}, 0)$. To find the interval of convergence we need to plug at the endpoints. Plug $x = -\frac{1}{2}$ to find

$$\sum_{n=1}^{\infty} \frac{4^n(-\frac{1}{2}+\frac{1}{4})^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which is absolutely convergent by comparing with a p-series with $p = 2$. Plugging $x = 0$ gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges by the p-series test. So the interval of convergence is $[-\frac{1}{2}, 0]$.