VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Test 2 – Solutions

NAME: Solutions.

Directions. This exam contains six questions and an extra credit question. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you need to use a theorem that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (10 pts)	
2 (15 pts)	
3 (15 pts)	
4 (25 pts)	
5 (15 pts)	
6 (20 pts)	
Extra Credit (05 pts)	
TOTAL (100 pts)	

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. (10 pts) Below, all roots of the characteristic equation for a constant coefficient homogeneous linear differential equation are given. State the order of the differential equation and find its general solution.

(a) $\lambda = 2, \lambda = 1, \lambda = -1$ (twice).

(b) $\lambda = -1 + 2i$ (twice), $\lambda = -1 - 2i$ (twice), $\lambda = 0$ (three times).

Solution 1. (a) Order 4. $x(t) = c_1 e^{2t} + c_2 e^t + c_3 e^{-t} + c_4 t e^{-t}$. (b) Order 7. $x(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + c_3 t e^{-t}\cos(2t) + c_4 t e^{-t}\sin(2t) + c_5 + c_6 t + c_7 t^2$.

Question 2. (15 pts)

(a) State the definition of a function of exponential order $\alpha > 0$.

(b) Determine whether the given function is of exponential order $\alpha > 0$.

(b1) $f(t) = e^{t^2}$.

(b2) $f(t) = \dfrac{1}{t}$.

(c) What is the relation between a function of exponential order α and the Laplace transform?

Solution 2. (a) f is said to be of exponential order $\alpha > 0$ if there exist positive constants T and M such that

$$|f(t)| \leq Me^{\alpha t}, \text{ for all } t \geq T.$$

(b1) For any $\alpha > 0$, we find

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t^2-\alpha t} = \infty,$$

hence e^{t^2} is not of exponential order α.

(b2) For $t \geq 1$ we have $\frac{1}{t} \leq 1$ and $e^t \geq 1$ thus

$$\left|\frac{1}{t}\right| \leq e^t \text{ for } t \geq 1$$

and we conclude that $\frac{1}{t}$ is of exponential order α.

(c) A function that is piecewise continuous and of exponential order $\alpha > 0$ on $[0, \infty)$ admits a Laplace transform.

Question 3. (15 pts) Find $\mathscr{L}^{-1}\{F\}$. You do not need to determine the constants of the partial fractions.

(a) $F(s) = \dfrac{s^2+9s+2}{(s^2-2s+1)(s+3)}$.

(b) $F(s) = \ln((s^2+4)(s+1))$.

Solution 3. (a) Write

$$\begin{aligned}\frac{s^2+9s+2}{(s^2-2s+1)(s+3)} &= \frac{s^2+9s+2}{(s-1)^2(s+3)}\\ &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3},\end{aligned}$$

so that

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{s^2+9s+2}{(s^2-2s+1)(s+3)}\} &= A\mathscr{L}^{-1}\{\frac{1}{s-1}\} + B\mathscr{L}^{-1}\{\frac{1}{(s-1)^2}\} + C\mathscr{L}^{-1}\{\frac{1}{s+3}\}\\ &= Ae^t + Bte^t + Ce^{-3t}.\end{aligned}$$

(b) Write $F(s) = \ln((s^2+4)(s+1)) = \ln(s^2+4) + \ln(s+1)$ and compute

$$\begin{aligned}\frac{dF(s)}{ds} &= \frac{d}{ds}\ln(s^2+4) + \frac{d}{ds}\ln(s+1)\\ &= \frac{2s}{s^2+4} + \frac{1}{s+1}.\end{aligned}$$

Then

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{dF(s)}{ds}\} &= 2\mathscr{L}^{-1}\{\frac{s}{s^2+4}\} + \mathscr{L}^{-1}\{\frac{1}{s+1}\}\\ &= 2\cos(2t) + e^{-t}.\end{aligned}$$

Since $tf(t) = -\mathscr{L}^{-1}\{\frac{dF}{ds}\}$, we find

$$\mathscr{L}^{-1}\{\ln((s^2+4)(s+1))\} = -\frac{1}{t}(2\cos(2t) + e^{-t}).$$

Question 4. (25 pts) Solve the given initial value problem using the method of Laplace transforms. You do not need to determine the constants of the partial fractions.

(a) $x'' - 4x' + 5x = 4e^{3t}$, $x(0) = 2$, $x'(0) = 7$.

(b) $x'' - 2x' + x = 6t - 2$, $x(-1) = 3$, $x'(-1) = 7$.

(c) $x'' + x = \delta(t - \frac{\pi}{2})$, $x(0) = 0$, $x'(0) = 1$.

(d) $x'' - x = f(t)$, $x(0) = 1$, $x'(0) = 2$,
where $f(t)$ is given by

$$f(t) = \begin{cases} 1, & t < 3, \\ t, & t > 3. \end{cases}$$

Solution 4. (a) Taking the Laplace transform,

$$s^2X(s) - 2s - 7 - 4(sX(s) - 2) + 5X(s) = \frac{4}{s-3},$$

thus

$$\begin{aligned} X(s) &= \frac{2s^2 - 7s + 7}{(s^2 - 4s + 5)(s-3)} = \frac{2s^2 - 7s + 7}{((s-2)^2 + 1)(s-3)} \\ &= \frac{A}{s-3} + \frac{B(s-2) + C}{(s-2)^2 + 1} \\ &= \frac{A}{s-3} + \frac{B(s-2)}{(s-2)^2 + 1} + \frac{C}{(s-2)^2 + 1}. \end{aligned}$$

Thus, taking the inverse Laplace transform,

$$x(t) = Ae^{3t} + Be^{2t}\cos t + Ce^{2t}\sin t.$$

(b) Define $w(t)$ by $w(t) = x(t-1)$, so that $w'(t) = x'(t-1)$, $w''(t) = x''(t-1)$, $w(0) = 3$, and $w'(0) = 7$. Plugging $t - 1$ for t into the equation gives

$$x''(t-1) - 2x'(t-1) + x(t-1) = 6(t-1) - 2 = 6t - 8,$$

thus

$$w''(t) - 2w'(t) + w(t-1) = 6t - 8.$$

Taking the Laplace transform gives

$$s^2W(s) - 3s - 7 - 2(W(s) - 3) + W(s) = \frac{6}{s^2} - \frac{8}{s},$$

or

$$\begin{aligned} W(s) &= \frac{3s^3 + s^2 - 8s + 6}{s^2(s^2 - 2s + 1)} = \frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}. \end{aligned}$$

The inverse Laplace transform gives

$$w(t) = A + Bt + Ce^t + Dte^t.$$

Since $x(t) = w(t+1)$:

$$x(t) = A + B(t+1) + Ce^{t+1} + D(t+1)e^{t+1}.$$

(c) Taking the Laplace transform

$$s^2X(s) - s0 - 1 + X(s) = e^{-\frac{\pi}{2}s},$$

or

$$X(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}.$$

Taking the inverse Laplace transform gives

$$y(t) = \sin t + \sin(t - \frac{\pi}{2})u(t - \frac{\pi}{2}).$$

(d) Write $f(t)$ as $f(t) = \Pi_{0,3}(t) + tu(t-3)$. Taking the Laplace transform and recalling that $Pi_{0,3}(t) = u(t) - u(t-3)$, we find

$$s^2X(s) - s - 2 - X(s) = \frac{1}{s} - e^{-3t}s + \mathscr{L}\{tu(t-3)\}.$$

To compute the last term, write $\mathscr{L}\{tu(t-3)\} = \mathscr{L}\{(t-3+3)u(t-3)\} = \mathscr{L}\{z(t-3)u(t-3)\} = e^{-3s}Z(s)$, where $z(t) = t+3$, so that $Z(s) = (3s+1)/s^2$. Hence

$$\begin{aligned} X(s) &= \frac{s^2+2s+1}{s(s-1)(s+1)} + e^{-3s}\frac{2s+1}{s(s-1)(s+1)} = \frac{s+1}{s(s-1)} + e^{-3s}\frac{2s+1}{s^2(s-1)(s+1)} \\ &= \frac{A}{s} + \frac{B}{s-1} + e^{-3s}(\frac{C}{s} + \frac{D}{s^2} + \frac{E}{s-1} + \frac{F}{s+1}). \end{aligned}$$

The inverse Laplace transform gives

$$x(t) = A + Be^t + Cu(t-3) + D(t-3)u(t-3) + Ee^{t-3}u(t-3) + Fe^{-(t-3)}u(t-3).$$

Question 5. (15 pts) Solve the following integro-differential equation. You do not need to determine the constants of the partial fractions.

$$
\begin{aligned}
y'(t) &= 1 - \int_0^t y(t-\tau)e^{-2\tau}\, d\tau, \\
y(0) &= 1.
\end{aligned}
$$

Solution 5. Write the equation as

$$y'(t) = 1 - y(t) * e^{-2t}.$$

Taking the Laplace transform

$$sY(s) - 1 = \frac{1}{s} - \frac{1}{s+2}Y(s),$$

thus

$$Y(s) = \frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}.$$

Taking the inverse Laplace transform,

$$y(t) = A + Be^{-t}.$$

Question 6. (20 pts)

(a) State the properties that characterize $\delta(t-a)$.

(b) Compute the Laplace transform of $\delta(t-a)$, $a > 0$.

(c) For $\varepsilon > 0$, consider the functions $f_\varepsilon(t)$ defined by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

What is the relation between $f_\varepsilon(t)$ and $\delta(t)$?

(d) Using the functions $f_\varepsilon(t)$ from part (c), show that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0),$$

for any continuous function $f(t)$.

Solution 6. (a) The properties are

$$\delta(t-a) = \begin{cases} \infty, & t = a, \\ 0, & t \neq a, \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)\delta(t-a)f(t) = f(a),$$

for any continuous function $f(t)$.

(b) Using the properties stated in (a):

$$\mathscr{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.$$

(c) Taking the limit $\varepsilon \to 0^+$ we find

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0, \end{cases}$$

which is agrees with the first property stated in (a) (for $a = 0$). Thus, we think of $\delta(t)$ as the limit of $f_\varepsilon(t)$ when $\varepsilon \to 0^+$, although this limit is not a function.

(d) Note that $\int_{-\infty}^{\infty} f_\varepsilon(t)\,dt = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{1}{\varepsilon}\,dt = 1$ for any $\varepsilon > 0$. Write

$$\int_{-\infty}^{\infty} \delta(t)f(t)\,dt = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} f_\varepsilon(t)f(t)\,dt = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(t)\,dt.$$

Because $f(t)$ is continuous, it attains a maximum and a minimum on the closed interval $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ at some points t_M and t_m, respectively. Thus

$$f(t_m) = \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(t_m)\,dt \leq \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(t)\,dt \leq \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(t_M)\,dt = f(t_M).$$

Since $t_m \to 0$ and $t_M \to 0$ as $\varepsilon \to 0^+$ and f is continuous, we have $\lim_{\varepsilon\to 0^+} f(t_m) = \lim_{\varepsilon \to 0^+} f(t_M) = f(0)$, and we obtain the result by the squeeze theorem.

Extra credit. (05 pts) If $\mathscr{L}\{f\} = F$ and $\mathscr{L}\{g\} = G$, prove that

$$\mathscr{L}\{f * g\} = FG.$$

Solution 7. Done in class. See the class notes.