There is another way of thinking of linear PDEs, as follows. Consider F_H as an <u>operator</u>. An operator is an operation that assigns to a given function another function. For instance, in the example $F_H(x,u,Du) = xu_x + yu_y + u$, F_H assigns to the function u the new function $xu_x + yu_y + u$. Similarly, in the example $F_H(x,u,Du) = u_x^2 + u_y$, F_H assigns to u the new function $u_x^2 + u_y$

<u>Remark</u>: Notice that differentiation and integration are also operators. For instance, $\frac{\partial}{\partial x^1}$ assigns to a given u the new function $\frac{\partial u}{\partial x^1}$. Similarly $\int dx$ assigns to a given $f(x)$ the new function $\int f(x)\,dx$.

<u>Remark</u>. If you think about, an operator is also a function (comment)

Thinking of F_H as an operator, we will write

$$F_H(x,u,\ldots,D^m u) = Pu$$

, where we think of P as an operator and of Pu as P applied to u (the same way we think of $\frac{\partial}{\partial x^1}$ and of $\frac{\partial u}{\partial x^1}$)

We call P <u>the operator</u> associated with the PDE

Then, $F(x, u, \dots, D^m u) = 0$ is a linear equation if and only if P is a linear operator, i.e.,

$$P(au + bv) = aPu + bPv$$ for any two functions u, v and constants a, b such that both sides are well defined.

(Exercise)

<u>Remark</u> Notice that the concept of an operator is more general than that of the homogeneous part of a PDE, and we can talk about operators without having a PDE (e.g., $\frac{\partial}{\partial x}$).

<u>Terminology</u> We sometimes use the term $\overbrace{\underline{\text{differential operator}}}^{\text{mention the order of } P}$ to emphasize that a certain operator is constructed out of derivatives. (as will be most operators in this course).

When an operator comes from a PDE ($P = F_H$), we call it the <u>operator associated</u> with the PDE.

A PDE $F(x, u, \dots, D^m u) = 0$ can equivalently be written as

$$Pu = f$$

where f is a known function (equal to $-f$ of before)

Prop Let $u_1, \dots, u_k$ be solutions to the homogeneous linear PDE

$$Pu = 0$$

Then, the function $v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$ is also a solution, where $c_1, \dots, c_k$ are constants.

Proof We need to show that $Pv = 0$. Compute

$$Pv = P(c_1 u_1 + c_2 u_2 + \dots + c_k u_k) \underset{\text{by linearity}}{=} c_1 \underbrace{Pu_1}_{=0} + c_2 \underbrace{Pu_2}_{=0} + \dots + c_k \underbrace{Pu_k}_{=0} = 0 + \dots + 0 = 0.$$

$\boxtimes$

Notice that the result is not true for non-homogeneous linear PDEs.

Prop A first order PDE $Pu = f$ for $u = u(x,y)$ is linear if and only if it can be written as

$$a u_x + b u_y + c u = f$$

where a, b, and c are known functions of x and y

Proof/ First, assume that the PDE is written as stated:

$a u_x + b u_y + cu = f$, and let's show that it is linear. In this case, $Pu = a u_x + b u_y + cu$, so

$$P(c_1 u + c_2 v) = a\frac{\partial}{\partial x}(c_1 u + c_2 v) + b\frac{\partial}{\partial y}(c_1 u + c_2 v) + c(c_1 u + c_2 v) = a Pu + b Pv.$$

Reciprocally, assume that the PDE is linear. We want to show that it can be written as stated.

By linearity $F_H(x,y,u,u_x,u_y) = \underbrace{F_x(x,y,u_x) + F_y(x,y,u_y)}_{F_1(x,y,Du)} + F_0(x,y,u)$ (F_x, F_y = not derivatives)

Since $F_x(x,y,u_x)$ is a linear function of u_x, we must have $F_x(x,y,u_x) = a(x,y)u_x$ and similarly $F_y(x,y) = b(x,y)u_y$, $F_0(x,y,u) = c(x,y)u$, for some functions a, b, c.

Furthermore, since this holds for any function u, if we choose $u(x,y) = 1$, $F_H(x,y,1,0,0) = c(x,y)$.

Then, choosing $u(x,y) = x$:

$$F_H(x,y,x,1,0) = a(x,y) + c(x,y)\cdot x \Rightarrow a(x,y) = F_H(x,y,x,1,0) - \overbrace{c(x,y)}^{\text{already known}} x$$

and similarly $b(x,y) = F_H(x,y,y,0,1) - c(x,y)\,y$.

$\square$

We can use similar ideas to give the precise form of P for equations of higher order and of more variables. For example, a linear second order PDE for $u = u(x,y)$ can be written as

$$a^{xx} u_{xx} + a^{yy} u_{xy} + a^{xy} u_{xy} + b^{x} u_{x} + b^{y} u_{y} + cu = f$$

where $a^{xx}, \ldots, c$ are functions of x and y.

A linear first order PDE for $u = u(x^1, \ldots, x^n)$ can be written as

$$a^1 \frac{\partial u}{\partial x^1} + a^2 \frac{\partial u}{\partial x^2} + \cdots + a^n \frac{\partial u}{\partial x^n} + bu = f$$

where $a^1, \ldots, b$ are functions of $(x^1, \ldots, x^n)$.

A linear second order PDE for $u = u(x^1, \ldots, x^n)$ can be written as

$$\sum_{i,j=1}^{n} a^{ij} \partial^2_{ij} u + \sum_{i=1}^{n} b^i \partial_i u + cu = f$$

where $\partial^2_{ij} = \frac{\partial}{\partial x^i \partial x_j}$, $\partial_i = \frac{\partial}{\partial x^i}$, and the coefficients are functions of $(x^1, \ldots, x^n)$

<u>Notation</u> For coordinates $(x^1, \dots, x^n)$, we set

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \partial^2_{ij} = \partial_{ij} = \frac{\partial}{\partial x^i \partial x^j}, \quad \partial^3_{ijk} = \partial_{ijk} = \frac{\partial}{\partial x^i \partial x^j \partial x^k}$$

We also sometimes write ∂_{x^i}, $\partial_{x^i x^j}$ etc.

We also adopt the sum convention: when there is a sum over indices $i, j, k, \dots$ etc ranging from 1 to n, and indices appear repeated in the sum, we omit the summation symbol and the sum is implicitly understood. Thus,

$$\sum_{i,j=1}^{n} a^{ij} \partial_{ij} = a^{ij} \partial_{ij}, \quad \sum_{i=1}^{n} b^i \partial_i = b^i \partial_i \quad \text{etc.}$$

Thus, a linear equation cannot have any non-linear function of u or its derivatives: $u \cdot u_x$, $\cos(u)$, u_y^2, etc. cannot be present.

Notice that the coordinates can have non-linear functions, eg

$$x^2 u_x + u = 0 \quad \text{is a linear eq.}$$

First order PDEs

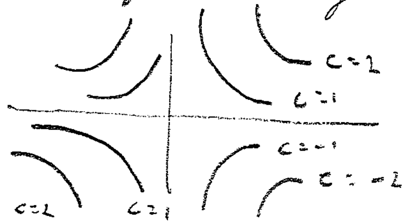
We will develop a general method for solving first order PDEs. In order to understand the intuition behind the method, let us work out some simple examples first.

Ex: Consider $au_x + bu_y = 0$, where a and b are constants not both zero.

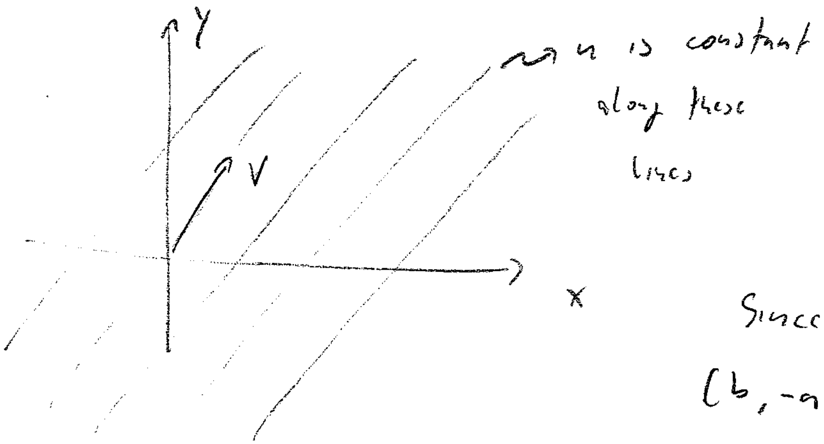
Notice that $au_x + bu_y$ is the directional derivative of u in the direction of the vector $v = (a,b)$: $D_v u = \nabla u \cdot v = \left(\frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j\right) \cdot (ai + bj) = au_x + bu_y$.

Thus, $au_x + bu_y = 0$ means that ∇u is orthogonal to v for any (x,y) where u is defined. But ∇u is orthogonal to the level curves of u: recall that the level curves of $g = g(x,y)$ are the sets $\Gamma_c = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = c\}$. So, for each $c \in \mathbb{R}$, Γ_c is a curve on the plane along which g is constant. For example, if $g(x,y) = xy$, then $xy = c \Rightarrow y = \frac{c}{x}$

or $x = 0$
$y = 0$



Therefore, we know that u must be constant in the direction of V:



The line through (x_0, y_0) in the direction of V is $(\, r(t) = tv + r_0 \,)$

$$(x, y) = t\,(a, b) + (x_0, y_0), \quad t \in \mathbb{R}$$

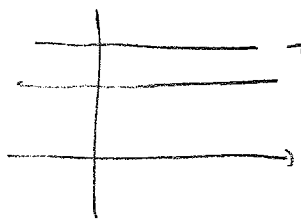
Since $(b, -a)$ is orthogonal to V, we have

$$(b, -a)\cdot(x, y) = (b, -a)\cdot(x_0, y_0) \Rightarrow \boxed{bx - ay = \text{constant}}$$

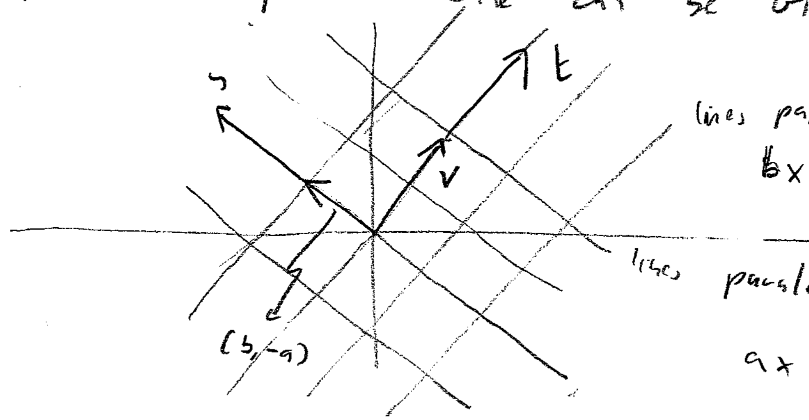
These are the lines in the direction of V (another way of seeing this: slope $= \frac{b}{a}$, $y = \frac{b}{a}x +$ intercept). Therefore $u(x,y)$ depends only on $bx - ay$, i.e., $u(x,y)$ is a function of $bx - ay$, i.e. $\boxed{u(x,y) = f(bx - ay)}$, where f is any function of one-variable. For instance, take $f(w) = w^2$, so $u(x,y) = (bx-ay)^2$ solves the equation. Verify $u_x(x,y) = 2(bx-ay)\cdot b$, $u_y(x,y) = 2(bx-ay)(-a)$, so

$$a\,u_x(x,y) + b\,u_y(x,y) = 2ab(bx-ay) - 2ab(bx-ay) = 0.$$

In the particular case when $b=0$, we have $a u_x = 0$. This means that u does not depend on x, so $u(x,y) = f(y)$ and u is constant along the lines $y = cte.$ Let's see that the previous case can be viewed as this one.

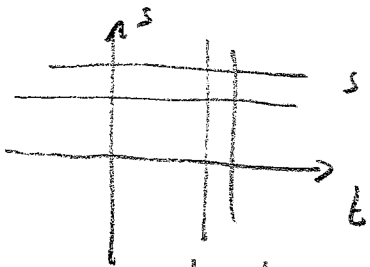


Idea:



let $s = bx - ay$, so the lines parallel to V correspond to $s = cte$

$t = ax + by$, so the lines parallel to $(b,-a)$ correspond to $t = cte$



Let $v(t,s) = u(x,y)$. (v is u written in the variables (t,s)). Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial v}{\partial t} a + \frac{\partial v}{\partial s} b$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial v}{\partial t} b - \frac{\partial v}{\partial s} a$$

So $\quad 0 = a u_x + b u_y = a^2 \frac{\partial v}{\partial t} + ab\frac{\partial v}{\partial s} + b^2 \frac{\partial v}{\partial t} - ab\frac{\partial v}{\partial s} = (a^2 + b^2)\frac{\partial v}{\partial t} \overset{a^2+b^2 \neq 0}{\Rightarrow} v = f(s)$

So $u(x,y) = v(t,s) = f(s) = f(bx - ay)$.

Ex: Solve $4u_x - 3u_y = 0$ with initial condition $u(0,y) = y^3$.

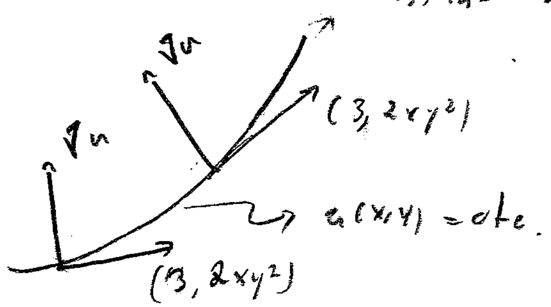
$u(x,y) = f(-3x - 4y)$. But $u(0,y) = f(-4y) = y^3$. Let $w = -4y$, so $y = -\frac{w}{4}$, and $f(-4y) = f(w) = y^3 = -\frac{w^3}{64}$, $f(w) = -\frac{w^3}{64}$ and $u(x,y) = \frac{(3x+4y)^3}{64}$.

Ex: Let us now investigate an equation where the coefficients are not constants. Consider (the linear PDE)

$$3u_x + 2xy^2 u_y = 0,$$ which we can write as $\nabla u \cdot (3, 2xy^2) = 0$.

Thus, for each point (x,y), $\nabla u(x,y)$ is orthogonal to the vector $(3, 2xy^2)$.

Thus, u is constant along the curves that have $(3, 2xy^2)$ as tangent vector.



Hence, we need to find such curves. If the curve is written $y = y(x)$, then the slope of the tangent vector is $\frac{dy}{dx}$, which has to equal $\frac{2xy^2}{3}$:

$$\frac{dy}{dx} = \frac{2xy^2}{3} \Rightarrow \text{equation for the curves along which } u \text{ is constant.}$$

$$\frac{dy}{y^2} = \frac{2}{3} x \, dx, \quad -\frac{1}{y^2} = \frac{1}{3}x^2 + C', \quad \text{so} \quad \frac{1}{3}x^2 + \frac{1}{y} = \text{constant} \quad (y \neq 0)$$

Therefore $u(x,y) = f\left(\frac{1}{3}x^2 + \frac{1}{y}\right)$.

Let's verify that this indeed solves the equation:

$$u_x(x,y) = f'\left(\frac{1}{3}x^2 + \frac{1}{y}\right) \cdot \frac{2}{3}x, \quad u_y(x,y) = f'\left(\frac{1}{3}x^2 + \frac{1}{y}\right)\left(-\frac{1}{y^2}\right)$$

$$3u_x + 2xy^2 u_y = 2x f'\left(\frac{1}{3}x^2 + \frac{1}{y}\right) + 2xy^2 f'\left(\frac{1}{3}x^2 + \frac{1}{y}\right)\left(-\frac{1}{y^2}\right) = 2x f'(\quad) - 2x f'(\quad) = 0.$$

<u>Upshot:</u> Given $a(x,y)u_x + b(x,y)u_y = 0$, the solution u will be given by $u(x,y) = f(A(x,y))$, where f is arbitrary and $A(x,y)$ is found by solving $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$ (so $A(x,y) =$ cote are solutions to $\frac{dy}{dx} = \frac{b}{a}$).

If further conditions are given, we can then determine f.

Notice that in the above examples we solve the PDE by solving certain ODEs. We will now generalize this idea for arbitrary first order PDEs.