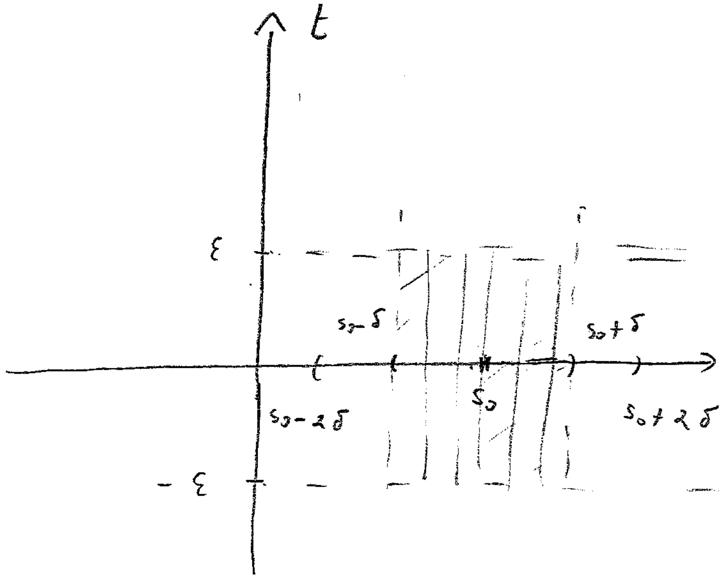
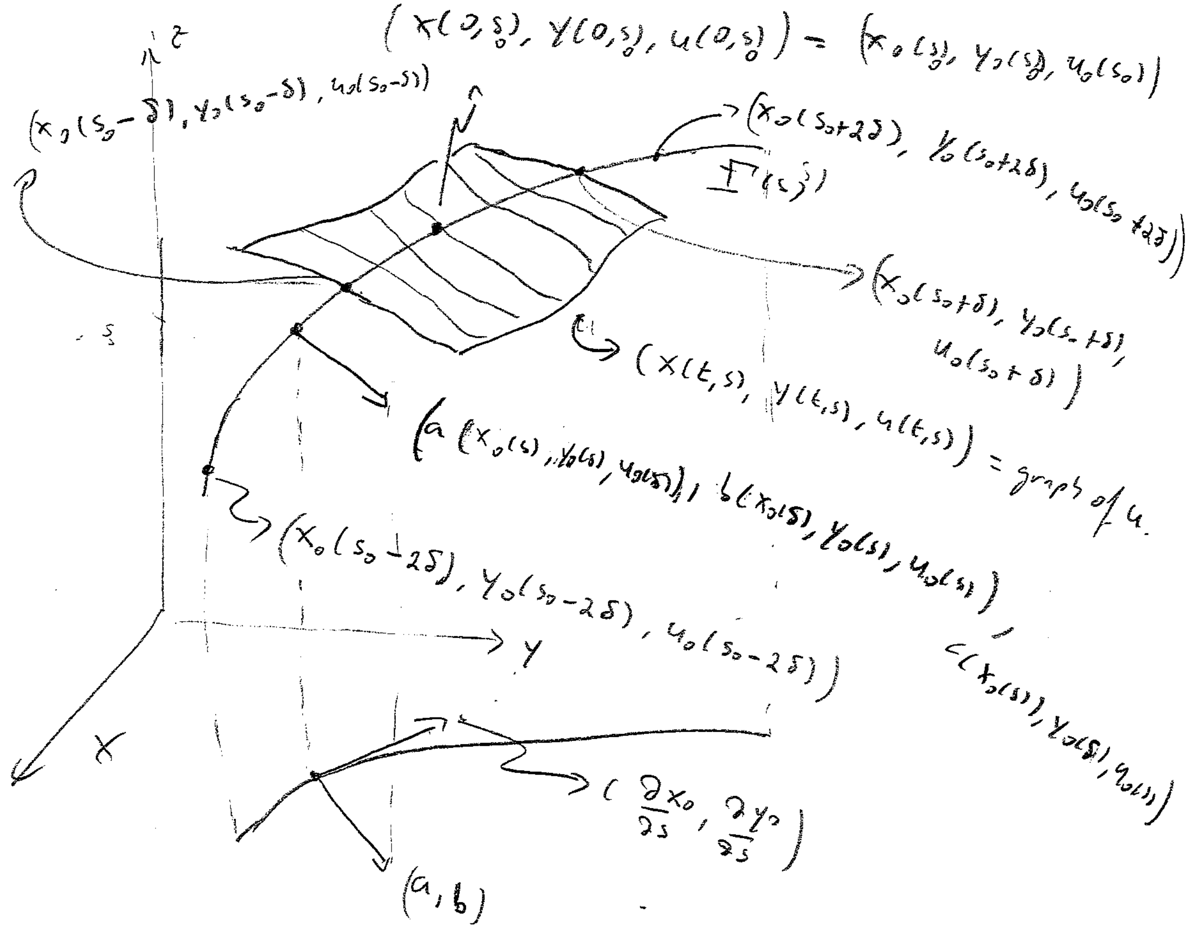




<u>Remarks</u>: • Notice that we are given an initial condition, so the statement about infinitely many solutions is not because of "undetermined functions".

• Notice that we can only guarantee a solution on the smaller interval $(s_0-\delta, s_0+\delta)$.

• Existence on $(-\varepsilon, \varepsilon)$ means that the integral surface exists on "both sides" of $\Gamma(s)$ (positive and negative "time") proof: By the existence and uniqueness theorem for ODEs, for each $s \in (s_0 - 2\delta, s_0 + 2\delta)$, there exists a characteristic curve $(x(t,s), y(t,s), u(t,s))$ starting at $(x_0(s), y_0(s), u_0(s))$ and existing on tan interval $(-\varepsilon_*(s), \varepsilon_*(s))$. Considering s on the small interval $(s_0-\delta, s_0+\delta)$ we can guarantee that there exists a $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq (-\varepsilon_*(s), \varepsilon_*(s))$ for all $s \in (s_0-\delta, s_0+\delta)$. The transversality condition implies that the family of characteristics $(x(t,s), y(t,s), u(t,s))$, $(t,s) \in (-\varepsilon, \varepsilon) \times (s_0-\delta, s_0+\delta)$ gives a parametrization of a smooth surface.

Next, we show that the constructed surface gives a solution to the PDE. Since $J(0,s) \neq 0$, by continuity and making ε smaller if necessary, we have that $J(t,s) \neq 0$ for $(t,s) \in (-\varepsilon, \varepsilon) \times (s_0-\delta, s_0+\delta)$. Thus, we can invert $x(t,s)$, $y(t,s)$ and set

$$u(x,y) = u(t(x,y), s(x,y))$$

Compute: $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x}$; $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial y} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial y}$

But
$$1 = \frac{\partial t}{\partial t} = \frac{\partial}{\partial t}(t(x,y)) = \frac{\partial t}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{=a} + \frac{\partial t}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{=b} = a\frac{\partial t}{\partial x} + b\frac{\partial t}{\partial y}$$

$$0 = \frac{\partial s}{\partial t} = \frac{\partial}{\partial t}(s(x,y)) = \frac{\partial s}{\partial x}\overbrace{\frac{\partial x}{\partial t}} + \frac{\partial s}{\partial y}\overbrace{\frac{\partial y}{\partial t}} = a\frac{\partial s}{\partial t} + b\frac{\partial s}{\partial y}$$

Thus, $a u_x + b u_y = a(u_t t_x + u_s s_x) + b(u_t t_y + u_s s_y) = u_t\underbrace{(a t_x + b t_y)}_{=1} + u_s\underbrace{(a s_x + b s_y)}_{=0}$

$a u_x + b u_y = u_t$: But $u_t = c$ by the characteristic equations, thus

$a u_x + b u_y = c$, showing existence.

We now move to prove uniqueness. Suppose we are given a solution $f = f(x,y)$, and let $(x(t,s), y(t,s), u(t,s))$ be a characteristic curve. We must have

$$f(x(0,s), y(0,s), u(0,s)) = u(0,s)$$ since by assumption f solves (*).

Set

$$F(t,s) := u(t,s) - f(x(t,s), y(t,s)).$$

Compute

$$F_t(t,s) = u_t(t,s) - f_x(x,y)\,\dot{x}(t,s) - f_y(x,y)\,\dot{y}(t,s)$$

$$= c(x,y,u) - f_x(x,y)\,a(x,y,u) - f_y(x,y)\,b(x,y,u) \quad (\text{use } u = F+f)$$

$$F_t(t,s) = c(x,y,F+f) - f_x(x,y)\,a(x,y,F+f) - f_y(x,y)\,b(x,y,F+f) \quad (**)$$

By the initial condition, $F(0,s) = 0$. But $F(t,s)=0$ is a solution to the ODE $(**)$.
(plug $F=0$: $af_x+bf_y = c$ which is satisfied since f is a solution)

By the uniqueness result for ODEs, we conclude that

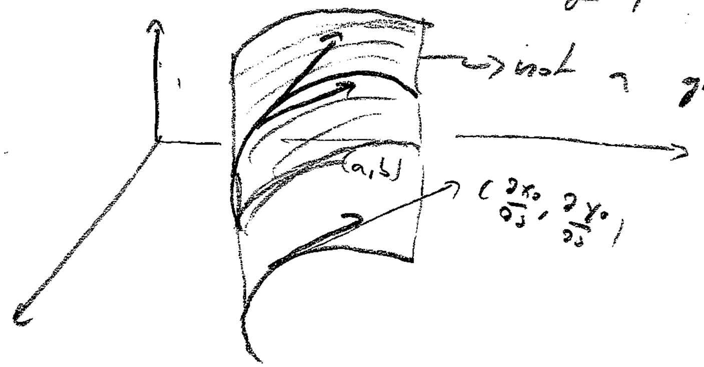
$$F(t,s) = u(t,s) - f(x(t,s), y(t,s)) = 0$$

and since this holds for any $s \in (s_0-\delta, s_0+\delta)$, we conclude that $f=u$, showing uniqueness.

Suppose now that the transversality condition does not hold on an interval of s values, so $\left(\frac{\partial x_0}{\partial s}, \frac{\partial y_0}{\partial s}\right)$ and $\left(a(x_0(s), y_0(s)), b(x_0(s), y_0(s))\right)$ are linearly dependent.

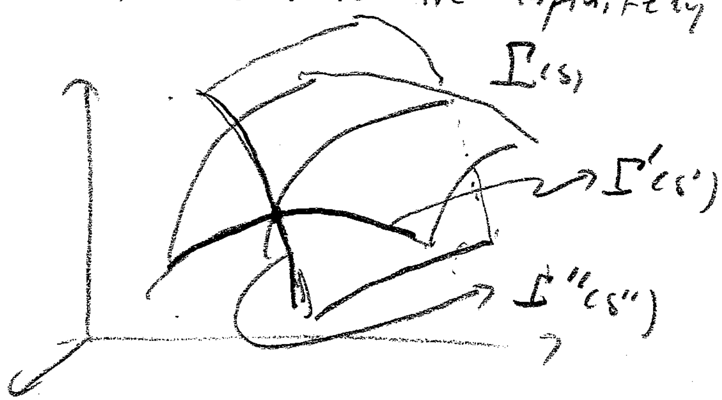
Then there are two possibilities: (i) a characteristic in the direction (a,b) starting at (x_0,y_0) agrees with $\Gamma(s)$, or (ii) it doesn't.

In case (ii), the characteristic and $\Gamma(s)$ have the same projection but don't agree, so the characteristic cannot belong to an integral surface, hence there is no solution. :



In case (i), we can construct a solution as follows. Pick $(x_0, y_0, u_0) \in \Gamma(s)$, let $\Gamma'(s')$ be a curve through (x_0, y_0, u_0) transversal to $\Gamma(s)$. By the existence part of the theorem, we can construct a parametrized surface S', which gives a solution to (*) (S will contain $\Gamma(s)$ since now $\Gamma(s)$ is a characteristic). This works for any choice of transversal $\Gamma'(s')$, thus there are infinitely many solutions.

$\boxtimes$



Remarks

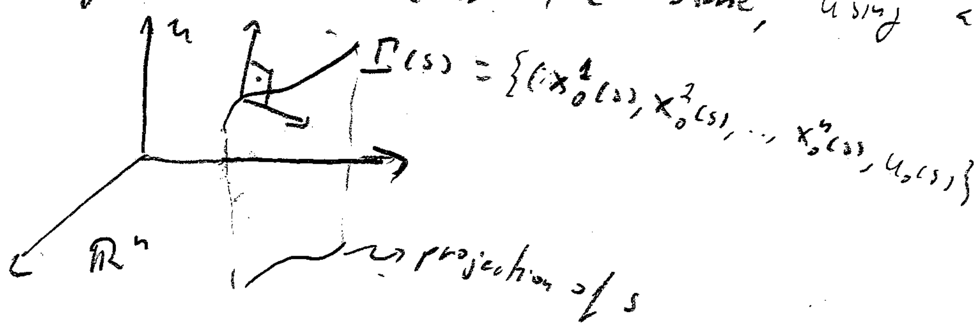
- The method of characteristics can be adapted to deal with "fully non-linear equations", i.e, non-linear equations that are not quasi-linear.

Ex: $u_x^2 + u_y^2 = 1$

$u_x u_x = a(x, y, u, u_x) u_x$

- The method can also be applied for functions of n variables $u = u(x^1, \dots, x^n)$.

The basic geometric idea is the same, using a non-transversality condition



Our previous experience with first order PDEs shows that we need to be more precise about the domain of definition of a PDE and its solution.

Def. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a <u>domain</u> in $\mathbb{R}^n$ (ie, an open and connected set). A PDE of order m for a function $u = u(x^1, \dots, x^n)$ in $\mathcal{U}$ is an equation

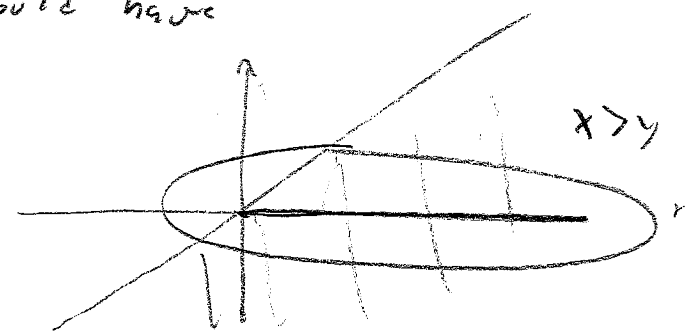
$$F(x, u, Du, \dots, D^m u) = 0. \quad (*)$$

A <u>solution</u> to the PDE is a function defined on $\mathcal{U}$ which satisfies $(*)$. We sometimes use the terminology <u>global solution</u> for a function that satisfies $(*)$ on all of $\mathcal{U}$, and <u>local solution</u> when a function satisfies $(*)$ only in part of $\mathcal{U}$.

<u>Ex:</u> Solve $u_x + u_y = 0$ in $\mathbb{R}^2$. We verify that $u(x,y) = \cos\sqrt{x-y}$ solves the equation. This solution is local because it is defined only on $\{(x,y) \mid x > y\}$ (notice that we cannot includ $x=y$ because u_x, u_y, blow-up there). On the other hand, if we take $\mathcal{U} = \{(x,y) \mid x > y\}$ then this solution is global in $\mathcal{U}$.

Remark The distinction global vs local solutions becomes important when we have an I.C. For example, we could have

$$\begin{cases} u_x + u_y = 0 & \text{in } \mathbb{R}^2 \\ u(x,0) = \cos\sqrt{x}, & x > 0 \end{cases}$$



The function $u(x,y) = \cos\sqrt{x-y}$ is thus a local solution as it exists only in $\{(x,y) \mid x > y\}$, i.e., in a neighborhood of the initial region $\{y = 0\}$.

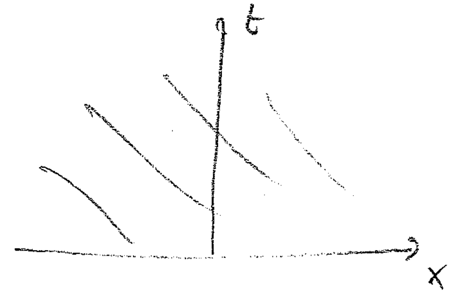
Notation Let u be a function defined in a domain $\mathcal{U} \subseteq \mathbb{R}^n$. We write

$$u \in C^k(\mathcal{U}) \quad \text{or simply} \quad u \in C^k \text{ when } \mathcal{U} \text{ is understood}$$

to mean that all derivatives of order k of u exist and are continuous in $\mathcal{U}$. If all derivatives of u exist, we write $u \in C^\infty$ and say that u is smooth. (so when we write $u \in C^k$, we can possibly have $k = \infty$).

The wave equation

We will study the one-dimensional wave equation:



$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad (x,t) \in (-\infty,\infty) \times (0,\infty)$$

Consider the change of variables $\alpha = x + ct$, $\beta = x - ct$ and set $v(\alpha,\beta) = u(x,t)$,

i.e., $u(x,t) = v(\alpha(x,t), \beta(x,t))$. Then

$$u_t = v_\alpha \frac{\partial \alpha}{\partial t} + v_\beta \frac{\partial \beta}{\partial t} = c\, v_\alpha - c\, v_\beta ,$$

$$u_{tt} = c\, \partial_t (v_\alpha - v_\beta) = c \left(v_{\alpha\alpha} \frac{\partial \alpha}{\partial t} + v_{\alpha\beta} \frac{\partial \beta}{\partial t} - v_{\beta\alpha} \frac{\partial \alpha}{\partial t} - v_{\beta\beta} \frac{\partial \beta}{\partial t} \right)$$

$$= c \left(v_{\alpha\alpha}\, c + v_{\alpha\beta} (-c) - v_{\beta\alpha}\, c - v_{\beta\beta} (-c) \right) = c^2 \left(v_{\alpha\alpha} - 2 v_{\alpha\beta} + v_{\beta\beta} \right)$$

And

$$u_x = v_\alpha \frac{\partial \alpha}{\partial x} + v_\beta \frac{\partial \beta}{\partial x} = v_\alpha + v_\beta$$

$$u_{xx} = \partial_x (v_\alpha + v_\beta) = v_{\alpha\alpha} \frac{\partial \alpha}{\partial x} + v_{\alpha\beta} \frac{\partial \beta}{\partial x} + v_{\beta\alpha} \frac{\partial \alpha}{\partial x} + v_{\beta\beta} \frac{\partial \beta}{\partial x} = v_{\alpha\alpha} + 2 v_{\alpha\beta} + v_{\beta\beta}$$

Hence

$$0 = u_{tt} - c^2 u_{xx} = c^2 \left(v_{\alpha\alpha} - 2 v_{\alpha\beta} + v_{\beta\beta} \right) - c^2 \left(v_{\alpha\alpha} + 2 v_{\alpha\beta} + v_{\beta\beta} \right)$$

$$= -c^2\, 4\, v_{\alpha\beta} \implies v_{\alpha\beta} = 0 \quad \text{since } c \neq 0$$

Write this as $\partial_\beta(\partial_\alpha v) = 0$. Thus, $\partial_\alpha v$ is independent of β, hence it is a function only of α: $\partial_\alpha v(\alpha,\beta) = f(\alpha)$, for some function f.

Integrating in α: $v(\alpha,\beta) = \int f(\alpha)\,d\alpha + G(\beta)$ where G is a function only of β. But $\int f(\alpha)\,d\alpha = F(\alpha)$ = function only of α. Thus

$$v(\alpha,\beta) = F(\alpha) + G(\beta).$$

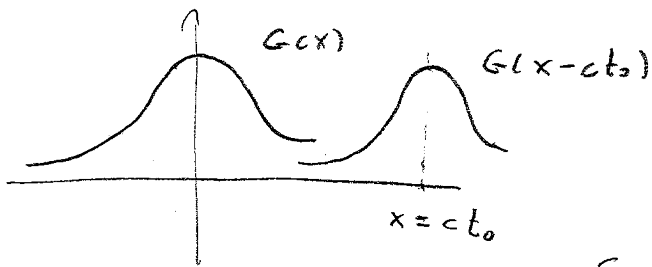
Now recall that $\alpha = x+ct$, $\beta = x-ct$, and $u(x,t) = v(\alpha,\beta)$, so

$$\boxed{u(x,t) = F(x+ct) + G(x-ct)} \quad (*)$$

This shows the following:

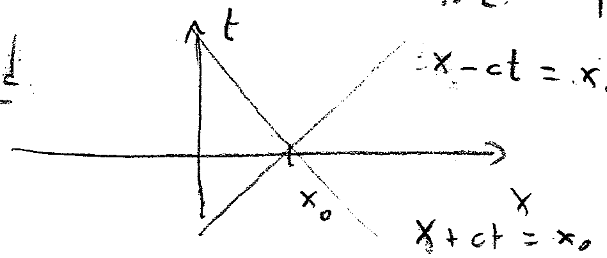
<u>Prop</u> If u is a solution of the one-dimensional wave equation, then there exist two functions $F, G \in C^2$ such that $(*)$ holds. Conversely, given any two functions $F, G \in C^2$, formula $(*)$ defines a solution to the one dimensional wave equation.

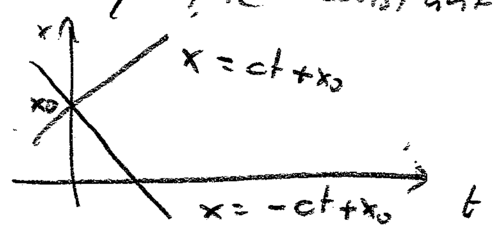
For any fixed $t_0 > 0$, the graph of $G(x - ct_0)$ is the same as that of $G(x)$, except that it is shifted to the right by the distance ct_0. Thus, as t moves, the graph of $G(x-ct)$ moves to the right. Since the distance travelled by the graph after time t is ct, the graph is moving at speed c.



Similarly, the graph of $F(x+ct)$ moves to the left at speed c.

Since $F(x+ct)$ and $G(x-ct)$ are solutions to the wave equation, we interpret them as waves moving (propagating) at speed c. G is called a <u>forward</u> wave and F a <u>backward</u> wave. Formula (*) above says that any solution to the wave equation in one dimension is a $\underbrace{\text{sum}}_{\text{"superposition"}}$ of a forward and a backward wave. That's why the constant c is the <u>wave speed</u>.





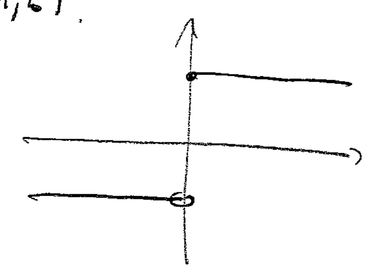
We derived (*) assuming that F and G are C^2. But given formula (*), we can ask if it is valid to consider functions that are not C^2. For instance, take $F(x) = G(x) = |x|$. Then (*) would give

$$u(x,t) = |x+ct| + |x-ct|$$

Since $|x|$ is C^2 except at $x=0$, we see that the above formula fails to give a solution to the wave equation only when $x = \pm ct$, so it seems that $|x+ct| + |x-ct|$ is "almost" a solution. This motivates the following:

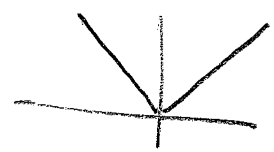
__Def__. A piecewise C^k function $f:(a,b) \to \mathbb{R}$ is a function that is C^k excep at a countable number of isolated points $\{x_i\}_{i=1}^{\infty} \subset (a,b)$.

__Ex__: $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ is a piecewise smooth (C^∞) function

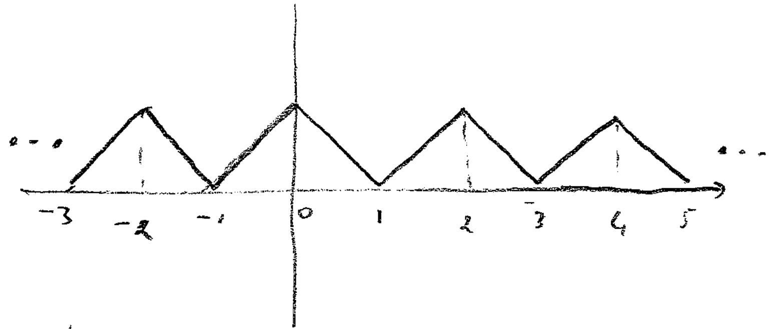


In this example $\{x_i\}_{i=1}^{\infty} = 1$ point $= 0$.

__Ex__: $f(x) = |x|$ is a piecewise smooth function

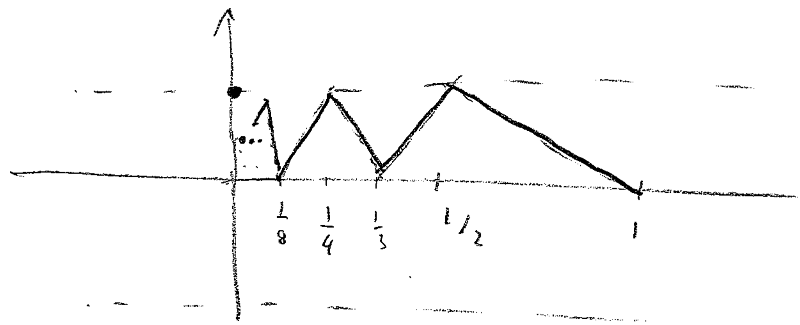


Ex: The function



is a piecewise smooth function. In this case $\{x_i\}_{i=1}^{\infty}$ = integers.

Ex: The function



is not piecewise smooth because in this case $\{x_i\}_{i=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$, and thus $\{0\}$ is not isolated.

Remarks: • The points $\{x_i\}_{i=1}^{\infty}$ where a piecewise C^k function fails to be C^k are sometimes called the singularities of the function.

• If $f:(a,b) \to \mathbb{R}$ is piecewise C^k and (a,b) is finite, then the number of singularities must be finite. But there can be infinitely many singularities if (a,b) is infinite (ex, $(a,b) = (-\infty,\infty)$, see above example).

Def. Let F and G be picewise C^2 functions. We call the function

$$u(x,t) = F(x+ct) + G(x-ct)$$

a generalized solution of the wave equation (aka a weak solution of the wave equation).

A generalized solution satisfies the wave equation except at the singularities.

A solution that contains no singularities (i.e., a solution in the "old" sense) is called a classical solution

Generalized solutions are important, e.g. in the study of shocks. The terminology generalized or weak solution may carry different meanings depending on the context.