**VANDERBILT UNIVERSITY**

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

*HW 5*

The problems below deal with solving PDEs via transformation methods. As we saw in class, many times these methods involve solving integrals that require advanced techniques. If you come across one of such integrals, you can quote the answer from a textbook, an online table of integrals, or a symbolic software like Mathematica or Maple. If you cannot find the answer in any of these sources, leave the integral indicated.

**Question 1.** Consider the following initial-value problem:

$$u_{tt} + 2du_t - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{1a}$$

$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{1b}$$

$$u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{1c}$$

where $d > 0$ is a constant. Equation (1) is known as the telegraph equation.

(a) Applying the Fourier transform in the spatial variable only, show that $\hat{u}$ solves the following problem:

$$\hat{u}_{tt} + 2d\hat{u}_t + |y|^2\hat{u} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{2a}$$

$$\hat{u} = \hat{g} \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{2b}$$

$$\hat{u}_t = \hat{h} \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{2c}$$

(b) Problem (2) is an ODE for $\hat{u}$ for each fixed $y$. Solve it by trying a solution of the form $\hat{u} = \beta e^{t\gamma}$, where $\beta$ and $\gamma$ can depend on $y$. You should find

$$\hat{u}(y,t) = \begin{cases} e^{-dt}(\beta_1(y)e^{\gamma(y)t} + \beta_2(y)e^{-\gamma(y)t}) & \text{if } |y| \leq d, \\ e^{-dt}(\beta_1(y)e^{i\delta(y)t} + \beta_2(y)e^{-i\delta(y)t}) & \text{if } |y| \geq d, \end{cases}$$

where $\gamma(y) = \sqrt{d^2 - |y|^2}$ with $|y| \leq d$, $\delta(y) = \sqrt{|y|^2 - d^2}$ with $|y| \geq d$, and $\beta_1$ and $\beta_2$ are selected such that

$$\hat{g}(y) = \beta_1(y) + \beta_2(y),$$

and

$$\hat{h}(y) = \begin{cases} \beta_1(y)(\gamma(y) - d) + \beta_2(y)(-\gamma(y) - d) & \text{if } |y| \leq d, \\ \beta_1(y)(i\delta(y) - d) + \beta_2(y)(-i\delta - d) & \text{if } |y| \geq d. \end{cases}$$

(c) Using the above, conclude that the solution is given by

$$u(x,t) = \frac{e^{-dt}}{\sqrt{2\pi}} \int_{|y|\leq d} (\beta_1(y)e^{ixy+\gamma(y)t} + \beta_2(y)e^{ixy-\gamma(y)t})\, dy$$
$$+ \frac{e^{-dt}}{\sqrt{2\pi}} \int_{|y|\geq d} (\beta_1(y)e^{i(xy+\delta(y)t)} + \beta_2(y)e^{i(xy-\delta(y)t)})\, dy.$$

(d) What happens when $t \to \infty$? Interpret your result and explain the meaning of the constant $d$.

**Question 2.** Consider the following initial-value problem:

$$iu_t + \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{3a}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}, \tag{3b}$$

where $i$ is the complex number $i^2 = -1$. Using the Fourier transform, show that the solution to (3) is

$$u(x,t) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} g(y)\, dy.$$

*Hint:* follow what we did in class for the heat equation.

**Remark:** Equation (3) is nothing else but the Schrödinger equation with the potential equal to zero and $\frac{\hbar}{2\mu} = 1$. This last condition can always be achieved by a suitable choice of units.

**Question 3.** Consider the following initial-value problem for the wave equation:

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{4a}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}, \tag{4b}$$

$$u_t = 0 \quad \text{on } \mathbb{R}^n \times \{t=0\}. \tag{4c}$$

Using the Fourier transform, show that the solution to (4) is

$$u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\hat{g}(y)}{2} (e^{i(x\cdot y + t|y|)} + e^{i(x\cdot y - t|y|)})\, dy,$$

where $i^2 = -1$. (Notice that the solution is real despite the presence of $i$.)
*Hint:* some ideas of problem (1) can be useful here.